

CLOSED UNITARY AND SIMILARITY ORBITS OF NORMAL OPERATORS IN PURELY INFINITE C*-ALGEBRAS

PAUL SKOUFRANIS

ABSTRACT. We will investigate the norm closure of the unitary and similarity orbits of normal operators in unital, simple, purely infinite C*-algebras. A complete classification will be given of when two normal operators are approximately unitarily equivalent in said algebras with trivial K_1 -group. Some upper and lower bounds for the distance between unitary orbits will be obtained. In addition, a complete characterization of when one normal operator is in the closed similarity orbit of another normal operator will be given for the Cuntz algebras \mathcal{O}_n (n finite) and type III factors with separable predual.

1. INTRODUCTION

Significant research has been performed in determining when two representations of a given C*-algebra are approximately unitarily equivalent. One example is Voiculescu's Theorem (see [Da4, Theorem II.5.8]) which provides various characterization of approximately unitarily equivalent representations of C*-algebras on separable Hilbert spaces. Furthermore, significant progress has been made in determining when two unital representations between unital, simple, purely infinite C*-algebras are approximately unitarily equivalent. In particular [KP, Proposition 0.6] shows any two unital *-homomorphisms from the Cuntz algebra \mathcal{O}_2 to a unital, simple, purely infinite C*-algebra are approximately unitarily equivalent and [KP, Theorem 1.13] shows any two injective, unital *-homomorphisms from a unital, separable, exact C*-algebra into \mathcal{O}_2 are approximately unitarily equivalent. Thus it is also of some interest to determine when two normal operators are approximately unitarily equivalent in a fixed C*-algebra \mathfrak{A} as this determines the classes of unitarily equivalent injective *-homomorphisms from singly generated abelian C*-algebra to \mathfrak{A} .

For the discussions in this paper, \mathfrak{A} will denote a unital C*-algebra, $\mathcal{U}(\mathfrak{A})$ will denote the unitary group of \mathfrak{A} , \mathfrak{A}^{-1} will denote the group of invertible elements of \mathfrak{A} , and \mathfrak{A}_0^{-1} will denote the connected component of the identity in \mathfrak{A}^{-1} . For a fixed C*-algebra \mathfrak{A} and an operator $A \in \mathfrak{A}$, let $\sigma(A)$ denote the spectrum of A in \mathfrak{A} , let

$$\mathcal{U}(A) := \{UAU^* \in \mathfrak{A} \mid U \in \mathcal{U}(\mathfrak{A})\},$$

and let

$$\mathcal{S}(A) := \{VAV^{-1} \in \mathfrak{A} \mid V \in \mathfrak{A}^{-1}\}.$$

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The set $\mathcal{U}(A)$ is called the unitary orbit of A in \mathfrak{A} and $\mathcal{S}(A)$ is called the similarity orbit of A in \mathfrak{A} .

Notice if $B \in \mathfrak{A}$ then $B \in \mathcal{U}(A)$ if and only if $A \in \mathcal{U}(B)$ and $B \in \mathcal{S}(A)$ if and only if $A \in \mathcal{S}(B)$. We will denote $B \in \mathcal{U}(A)$ by $A \sim_u B$ and we will denote $B \in \mathcal{S}(A)$ by $A \sim B$. Clearly \sim_u and \sim are equivalence relations.

We will use $\overline{\mathcal{U}(A)}$ and $\overline{\mathcal{S}(A)}$ to denote the norm closures in \mathfrak{A} of the unitary and similarity orbits of A respectively. Note if $B \in \overline{\mathcal{U}(A)}$ then $A \in \overline{\mathcal{U}(B)}$ and $B \in \overline{\mathcal{S}(A)}$. If $B \in \overline{\mathcal{U}(A)}$, we will say that A and B are approximately unitarily equivalent in \mathfrak{A} and will write $A \sim_{au} B$. Clearly \sim_{au} is an equivalence relation. Furthermore, if A is a normal operator and $A \sim_{au} B$, it is trivial to verify that B is a normal operator. However, if $B \in \overline{\mathcal{S}(A)}$, it is not clear that $A \in \overline{\mathcal{S}(B)}$ and B need not be normal if A is normal. However, if $B \in \overline{\mathcal{S}(A)}$ and $C \in \overline{\mathcal{S}(B)}$ then $C \in \overline{\mathcal{S}(A)}$.

It is an easy application of the semicontinuity of the spectrum to show that if $A, B \in \mathfrak{A}$ are such that $B \in \overline{\mathcal{S}(A)}$ then $\sigma(A) \subseteq \sigma(B)$ and $\sigma(A)$ intersects every connected component of $\sigma(B)$. Thus if $A, B \in \mathfrak{A}$ are approximately unitarily equivalent, $\sigma(A) = \sigma(B)$.

The concept of determining when two normal operators are approximately unitarily equivalent has been well-studied for the C^* -algebra of all bounded linear operators on a complex, separable, infinite dimensional Hilbert space.

Theorem 1.1 (see [Da4, Theorem II.4.4] for example). *Let $\mathcal{B}(\mathcal{H})$ be the C^* -algebra of all bounded linear operators on a complex, separable, infinite dimensional Hilbert space \mathcal{H} and let $N_1, N_2 \in \mathcal{B}(\mathcal{H})$ be normal operators. Then $N_1 \sim_{au} N_2$ if and only if $\text{rank}(\chi_U(N_1)) = \text{rank}(\chi_U(N_2))$ for all open subsets $U \subseteq \mathbb{C}$ (where $\chi_U(T)$ is the spectral projection of the normal operator T onto the open set U) if and only if*

- (1) $\sigma(N_1) = \sigma(N_2)$, and
- (2) $\dim(\ker(\lambda I_{\mathcal{H}} - N_1)) = \dim(\ker(\lambda I_{\mathcal{H}} - N_2))$ whenever $\lambda \in \sigma(N_1)$ is an isolated point.

The classification of when two normal operators are approximately unitarily equivalent in the Calkin algebra was completed in a famous paper due to Brown, Douglas, and Fillmore.

Theorem 1.2 ([BDF, Theorem 11.1]). *Let N_1 and N_2 be normal operators in the Calkin algebra. Then $N_1 \sim_{au} N_2$ if and only if $N_1 \sim_u N_2$ if and only if $\sigma(N_1) = \sigma(N_2)$ and the Fredholm index of $\lambda I - N_1$ and $\lambda I - N_2$ agree for all $\lambda \notin \sigma(N_1)$.*

More recently, the classification of when two normal operators are approximately unitarily equivalent in a von Neumann algebra was completed.

Theorem 1.3 ([Sh, Theorem 1.3]). *Let \mathfrak{M} be a von Neumann algebra of arbitrary (single) type and cardinality and let $N_1, N_2 \in \mathfrak{M}$ be normal operators. Then $N_1 \sim_{au} N_2$ if and only if $\chi_U(N_1)$ and $\chi_U(N_2)$ are Murray-von Neumann equivalent in \mathfrak{M} for all open subsets $U \subseteq \mathbb{C}$*

The closed similarity orbit of an arbitrary operator $T \in \mathcal{B}(\mathcal{H})$ has been also been determined. In particular, the following result for normal operators is easily stated.

Theorem 1.4 ([BH, Theorem 1]). *Let $\mathcal{B}(\mathcal{H})$ be the C^* -algebra of all bounded linear operators on a complex, separable, infinite dimensional Hilbert space \mathcal{H} . If $N, M \in \mathcal{B}(\mathcal{H})$ are normal operators with $\sigma(M) = \sigma_e(M)$, $N \in \overline{\mathcal{S}(M)}$ if and only if*

- (1) $\sigma(M) \subseteq \sigma(N)$ and $\sigma_e(M) \subseteq \sigma_e(N)$,
- (2) if $\lambda \in \sigma(N)$ is isolated, $\ker(\lambda I_{\mathcal{H}} - M)$ and $\ker(\lambda I_{\mathcal{H}} - N)$ have the same dimension, and
- (3) if $\lambda \in \sigma_e(N)$ is not isolated in $\sigma_e(N)$, the component of λ in $\sigma_e(N)$ contains some non-isolated point of $\sigma_e(M)$.

Note that a complete classification of the closed similarity orbit of an arbitrary bounded linear operator on a complex, infinite dimensional Hilbert space was announced in [AHV, Theorem 1] and a proof was given in [AFHV, Theorem 9.2]. Furthermore, an easy modification of the proof led to a complete classification of the closed similarity orbit of an arbitrary operator in the Calkin algebra (of which we state only the result for normal operators).

Theorem 1.5 ([AHV, Theorem 2], see [AFHV, Theorem 9.3] for a proof). *Let N and M be normal operators in the Calkin algebra. Then $N \in \overline{\mathcal{S}(M)}$ if and only if*

- (1) $\sigma_e(M) \subseteq \sigma_e(N)$,
- (2) each component of $\sigma_e(N)$ intersects $\sigma_e(M)$,
- (3) the Fredholm index of $\lambda I_{\mathfrak{A}} - M$ and $\lambda I_{\mathfrak{A}} - N$ agree for all $\lambda \notin \sigma_e(N)$, and
- (4) if $\lambda \in \sigma_e(N)$ is not isolated in $\sigma_e(N)$, the component of λ in $\sigma_e(N)$ contains some non-isolated point of $\sigma_e(M)$.

The purpose of this document is to examine when a normal operator is in the closed unitary or similarity orbit of another normal operator in a unital, simple, purely infinite C*-algebra. As such, if two normal operators N and M on a complex, separable, infinite dimensional Hilbert space are approximately unitarily equivalent and the C*-algebra generated by N and M is contained in a unital, simple, purely infinite C*-algebra \mathfrak{A} , it may be possible to select the unitary operators implementing the approximate unitary equivalence from \mathfrak{A} .

As the Calkin algebra is a unital, simple, purely infinite C*-algebra, these results are an attempt to generalize Theorem 1.2 and Theorem 1.5. However, due to technical restraints, a complete generalization has not been obtained. A complete characterization of approximate unitary equivalence of normal operators is obtained for unital, simple, purely infinite C*-algebras with trivial K_1 -group in Section 2. Section 3 then examines the distance between the unitary orbits of two normal operators in this setting. Finally Section 4 gives sufficient and necessary conditions for when one normal operator is contained the closed similarity orbit of another normal operator in \mathcal{O}_n , the Cuntz algebra generated by n isometries (with n finite), which enables an alternate proof to that given in [Sk, Theorem 2.8] of the classification of normal limits of nilpotent operators in \mathcal{O}_n .

2. CLOSED UNITARY ORBITS OF NORMAL OPERATORS

In this section we will completely classify when two normal operators in a unital, simple, purely infinite C*-algebra with trivial K_1 -group are approximately unitarily equivalent (see Corollary 2.16). The reasons for studying approximate unitary equivalence of normal operators in unital, simple, purely infinite C*-algebras is the K -theory of the projections of said algebras along with the following essential result due to Lin.

Theorem 2.1 ([Li, Theorem 4.4]). *Let \mathfrak{A} be a unital, simple, purely infinite C*-algebra and let $N \in \mathfrak{A}$ be a normal operator. Then N can be approximated by normal operators with finite spectra if and only if $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$.*

Using Lin's result and the following trivial technical detail, we can easily prove the following result for normal operators in a unital, simple, purely infinite C^* -algebra with trivial K_0 -group.

Lemma 2.2. *Let \mathfrak{A} be a C^* -algebra, let $N \in \mathfrak{A}$ be a normal operator, let U be an open subset of \mathbb{C} such that $U \cap \sigma(N) \neq \emptyset$, and let $(N_n)_{n \geq 1}$ be a sequence from normal operators of \mathfrak{A} such that $N = \lim_{n \rightarrow \infty} N_n$. Then there exists a $k \in \mathbb{N}$ such that $\sigma(N_n) \cap U \neq \emptyset$ for all $n \geq k$.*

Proof. Fix $\lambda \in U \cap \sigma(N)$. By Urysohn's Lemma there exists a continuous function f on \mathbb{C} such that $f|_{U^c} = 0$ yet $f(\lambda) = 1$. Note $f(N) = \lim_{n \rightarrow \infty} f(N_n)$ by standard functional calculus results. If $\sigma(N_n) \cap U = \emptyset$ for infinitely many n then $f(N_n) = 0$ for infinitely many n yet $f(N) \neq 0$ by construction. This is clearly a contradiction. \square

Proposition 2.3. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra such that $K_0(\mathfrak{A})$ is trivial. Let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that $\sigma(N_1) = \sigma(N_2)$ and $\lambda I_{\mathfrak{A}} - N_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1, 2\}$. Then $N_1 \sim_{au} N_2$.*

Proof. Since $K_0(\mathfrak{A}) = \{0\}$, all non-trivial projections are Murray-von Neumann equivalent. Thus any two normal operators with the same finite spectrum are unitarily equivalent.

By the assumption that $\lambda I_{\mathfrak{A}} - N_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1, 2\}$, N_1 and N_2 can be approximated by normal operators with finite spectrum by Theorem 2.1. By small perturbations using Lemma 2.2 and the semicontinuity of the spectrum, we can assume that N_1 and N_2 can be approximated arbitrarily well by normal operators with the same finite spectrum. Thus the result follows. \square

Note the condition $\lambda I_{\mathfrak{A}} - N_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1, 2\}$ holds when $\mathfrak{A}_0^{-1} = \mathfrak{A}^{-1}$. Moreover, for unital, simple, purely infinite C^* -algebras, $\mathfrak{A}_0^{-1} = \mathfrak{A}^{-1}$ if and only if $K_1(\mathfrak{A})$ is trivial (see [Cu, Theorem 1.9]).

If \mathcal{O}_2 is the Cuntz algebra generated by two isometries, $K_0(\mathcal{O}_2)$ and $K_1(\mathcal{O}_2)$ are trivial by [Cu, Theorem 3.7] and [Cu, Theorem 3.8] respectively. Thus Proposition 2.3 completely classifies when two normal operators in \mathcal{O}_2 are approximately unitarily equivalent.

Corollary 2.4. *Let $N, M \in \mathcal{O}_2$ be normal operators. Then $N \sim_{au} M$ if and only if $\sigma(N) = \sigma(M)$.*

With the above corollary, we note the following interesting result.

Corollary 2.5. *Let $\mathcal{B}(\mathcal{H})$ be the C^* -algebra of all bounded linear operators on a complex, separable, infinite dimensional Hilbert space \mathcal{H} and let $N_1, N_2 \in \mathcal{B}(\mathcal{H})$ be normal operators such that the C^* -algebra \mathfrak{A} generated by N_1 and N_2 is isomorphic to a C^* -subalgebra of $\mathcal{O}_2 \subseteq \mathcal{B}(\mathcal{H})$ (for example, when \mathfrak{A} is separable and exact by [KP, Theorem 2.8]). If N_1 and N_2 are approximately unitarily equivalent in $\mathcal{B}(\mathcal{H})$ then the unitaries implementing the approximate unitary equivalence may be taken from \mathcal{O}_2 .*

Note that the proof of Proposition 2.3 is easily modified to a more general setting. To see this, we recall the following definitions.

Definition 2.6. Let \mathfrak{A} be a unital C^* -algebra. We say that \mathfrak{A} has the finite normal property (property (FN)) if every normal operator in \mathfrak{A} is the limit of

normal operators from \mathfrak{A} with finite spectrum. We say that \mathfrak{A} has the weak finite normal property (property weak (FN)) if every normal operator $N \in \mathfrak{A}$ such that $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$ is the limit of normal operators from \mathfrak{A} with finite spectrum.

Corollary 2.7. *Let \mathfrak{A} be a unital C^* -algebra such that \mathfrak{A} has property weak (FN) and any two non-zero projections in \mathfrak{A} are Murray-von Neumann equivalent. If $N_1, N_2 \in \mathfrak{A}$ are two normal operators such that $\lambda I_{\mathfrak{A}} - N_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1, 2\}$ then $N_1 \sim_{au} N_2$ if and only if $\sigma(N) = \sigma(M)$.*

Corollary 2.8. *Let \mathfrak{A} be a unital C^* -algebra such that \mathfrak{A} has property (FN) and any two non-zero projections in \mathfrak{A} are Murray-von Neumann equivalent. If $N_1, N_2 \in \mathfrak{A}$ are two normal operators then $N_1 \sim_{au} N_2$ if and only if $\sigma(N) = \sigma(M)$.*

Corollary 2.9. *Let \mathfrak{M} be a type III factor and let $N_1, N_2 \in \mathfrak{M}$ be normal operators. Then $N_1 \sim_{au} N_2$ if and only if $\sigma(N_1) = \sigma(N_2)$.*

Of course Proposition 2.3 does not completely solve our problem in the case that $K_0(\mathfrak{A})$ is trivial and $K_1(\mathfrak{A})$ is non-trivial. We note the following is a necessary requirement.

Proposition 2.10. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that $N_1 \in \overline{\mathcal{S}(N_2)}$. Then $\lambda I_{\mathfrak{A}} - N_1$ and $\lambda I_{\mathfrak{A}} - N_2$ are in the same connected component of \mathfrak{A}^{-1} for all $\lambda \notin \sigma(N_1)$.*

Proof. Suppose $N_1 \in \overline{\mathcal{S}(N_2)}$ and $\lambda \notin \sigma(N_1)$. Then $\sigma(N_2) \subseteq \sigma(N_1)$ and there exists a sequence of invertible elements $V_n \in \mathfrak{A}$ such that

$$\lim_{n \rightarrow \infty} \|N_1 - V_n N_2 V_n^{-1}\| = 0.$$

Thus it is clear that

$$\lim_{n \rightarrow \infty} \|(\lambda I_{\mathfrak{A}} - N_1) - V_n(\lambda I_{\mathfrak{A}} - N_2)V_n^{-1}\| = 0.$$

Therefore, for sufficiently large n , $\lambda I_{\mathfrak{A}} - N_1$ and $V_n(\lambda I_{\mathfrak{A}} - N_2)V_n^{-1}$ are in the same connected component of \mathfrak{A}^{-1} . This implies

$$\begin{aligned} [\lambda I_{\mathfrak{A}} - N_1]_1 &= [V_n(\lambda I_{\mathfrak{A}} - N_2)V_n^{-1}]_1 \\ &= [V_n]_1 [\lambda I_{\mathfrak{A}} - N_2]_1 [V_n^{-1}]_1 \\ &= [\lambda I_{\mathfrak{A}} - N_2]_1. \end{aligned}$$

Therefore $\lambda I_{\mathfrak{A}} - N_1$ and $\lambda I_{\mathfrak{A}} - N_2$ are in the same connected component of \mathfrak{A}^{-1} by [Cu, Theorem 1.9]. \square

Our next task is to study the case where $K_0(\mathfrak{A})$ is non-trivial. The Cuntz algebras, \mathcal{O}_n , generated by $n \in \mathbb{N} \cup \{\infty\}$ isometries (where $K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1}$ and $K_1(\mathcal{O}_n)$ is trivial by [Cu, Theorem 3.7] and [Cu, Theorem 3.8] respectively) are excellent examples of unital, simple, purely infinite C^* -algebra where $K_0(\mathfrak{A})$ is non-trivial. We begin with the case that our two normal operators have the same connected spectrum. The following lemma contains the main argument of the general result and is motivated by the proof of [Sk, Theorem 2.8].

Lemma 2.11. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators. Suppose that $\sigma(N_1) = \sigma(N_2)$, $\sigma(N_1)$ is connected, and $\lambda I_{\mathfrak{A}} - N_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1, 2\}$. Then $N_1 \sim_{au} N_2$.*

Proof. We shall prove this result in the case that $\sigma(N_1) = \sigma(N_2) = [0, 1]$ and provide an outline the general case.

Suppose $\sigma(N_1) = [0, 1] = \sigma(N_2)$. Let $\epsilon > 0$ and choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. By Theorem 2.1 (or the fact that unital, simple, purely infinite C*-algebras have real rank zero (see [Da4, Theorem V.7.4])), by Lemma 2.2, by the semicontinuity of the spectrum, and by perturbing eigenvalues, there exists two collections of non-zero, pairwise orthogonal projections

$$\left\{ P_j^{(1)} \right\}_{j=0}^n \quad \text{and} \quad \left\{ P_j^{(2)} \right\}_{j=0}^n$$

in \mathfrak{A} such that

$$\sum_{j=0}^n P_j^{(q)} = I_{\mathfrak{A}} \quad \text{and} \quad \left\| N_q - \sum_{j=0}^n \frac{j}{n} P_j^{(q)} \right\| < 2\epsilon$$

for all $q \in \{1, 2\}$. Unfortunately, as $K_0(\mathfrak{A})$ may be non-trivial, we cannot apply the same idea as Proposition 2.3. The idea of the proof is to apply a ‘back and forth’ argument to produce a unitary that intertwines the approximations of N_1 and N_2 .

To begin, we note that since \mathfrak{A} is a unital, simple, purely infinite C*-algebra, $P_0^{(1)}$ is Murray-von Neumann equivalent to a proper subprojection of $P_0^{(2)}$. Thus we can write $P_0^{(2)} = Q_0^{(2)} + R_0^{(2)}$ where $Q_0^{(2)}$ and $R_0^{(2)}$ are non-zero orthogonal projections in \mathfrak{A} such that $Q_0^{(2)}$ and $P_0^{(1)}$ are Murray-von Neumann equivalent. Furthermore $R_0^{(2)}$ is Murray-von Neumann equivalent to a proper subprojection of $P_1^{(1)}$. Thus we can write $P_1^{(1)} = Q_1^{(1)} + R_1^{(1)}$ where $Q_1^{(1)}$ and $R_1^{(1)}$ are non-zero orthogonal projections in \mathfrak{A} such that $Q_1^{(1)}$ and $R_0^{(2)}$ are Murray-von Neumann equivalent.

For notional purposes, let $Q_0^{(1)} := 0$, $R_0^{(1)} := P_0^{(1)}$, $Q_n^{(2)} := P_n^{(2)}$, and $R_n^{(2)} := 0$. By repeating this procedure (using $R_1^{(1)}$ in place of $P_0^{(1)}$), we obtain sets of non-zero, pairwise orthogonal projections

$$\left\{ Q_j^{(1)}, R_j^{(1)} \right\}_{j=1}^n \quad \text{and} \quad \left\{ Q_j^{(2)}, R_j^{(2)} \right\}_{j=0}^{n-1}$$

such that $P_j^{(q)} = Q_j^{(q)} + R_j^{(q)}$ for all $j \in \{0, \dots, n\}$ and $q \in \{1, 2\}$, $R_j^{(2)}$ is Murray-von Neumann equivalent to $Q_{j+1}^{(1)}$ for all $j \in \{0, \dots, n-1\}$, and $R_j^{(1)}$ is Murray-von Neumann equivalent to $Q_j^{(2)}$ for all $j \in \{0, \dots, n-1\}$. Since

$$I_{\mathfrak{A}} = \sum_{j=0}^n Q_j^{(1)} + R_j^{(1)} = \sum_{j=0}^n Q_j^{(2)} + R_j^{(2)}, \quad (*)$$

we note that

$$\begin{aligned} \left[R_n^{(1)} \right]_0 &= \left[I_{\mathfrak{A}} \right]_0 - \sum_{j=1}^n \left[Q_j^{(1)} \right]_0 - \sum_{j=0}^{n-1} \left[R_j^{(1)} \right]_0 \\ &= \left[I_{\mathfrak{A}} \right]_0 - \sum_{j=1}^n \left[R_{j-1}^{(2)} \right]_0 - \sum_{j=0}^{n-1} \left[Q_j^{(2)} \right]_0 \\ &= \left[Q_n^{(2)} \right]_0. \end{aligned}$$

Hence $R_n^{(1)}$ and $Q_n^{(2)}$ are Murray-von Neumann equivalent by [Cu, Theorem 1.4].

Let $\{V_j\}_{j=0}^n \cup \{W_j\}_{j=0}^{n-1}$ be partial isometries in \mathfrak{A} such that $V_j^* V_j = R_j^{(1)}$ and $V_j V_j^* = Q_j^{(2)}$ for all $j \in \{0, \dots, n\}$, and $W_j^* W_j = Q_{j+1}^{(1)}$ and $W_j W_j^* = R_j^{(2)}$ for all

$j \in \{0, \dots, n-1\}$. Hence $(*)$ implies that

$$U := \sum_{j=0}^n V_j + \sum_{j=0}^{n-1} W_j$$

is a unitary operator in \mathfrak{A} . Moreover

$$\begin{aligned} U^* \left(\sum_{j=0}^n \frac{j}{n} P_j^{(2)} \right) U &= U^* \left(\sum_{j=0}^n \frac{j}{n} Q_j^{(2)} + \sum_{j=0}^n \frac{j}{n} R_j^{(2)} \right) U \\ &= \sum_{j=0}^n \frac{j}{n} R_j^{(1)} + \sum_{j=0}^{n-1} \frac{j}{n} Q_{j+1}^{(1)}. \end{aligned}$$

Hence, since

$$\sum_{j=0}^n \frac{j}{n} P_j^{(1)} = \sum_{j=0}^n \frac{j}{n} Q_j^{(1)} + \sum_{j=0}^n \frac{j}{n} R_j^{(1)},$$

we obtain that

$$\|N_1 - U^* N_2 U\| \leq \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $N_1 \sim_{au} N_2$.

To complete the general case, we will use a technique similar to that used in the proof of [Sk, Theorem 2.8]. To begin, let N_1 and N_2 be as in the statement of the lemma. Fix $\epsilon > 0$ and for each $(n, m) \in \mathbb{Z}^2$ let

$$B_{n,m} := \left(\epsilon n - \frac{\epsilon}{2}, \epsilon n + \frac{\epsilon}{2} \right] + i \left(\epsilon m - \frac{\epsilon}{2}, \epsilon m + \frac{\epsilon}{2} \right] \subseteq \mathbb{C}.$$

Thus the sets $B_{n,m}$ partition the complex plane into a grid with side-lengths ϵ .

For each $(n, m) \in \mathbb{Z}^2$ we label the box $B_{n,m}$ relevant if $\sigma(N_1) \cap B_{n,m} \neq \emptyset$ and we will say two boxes are adjacent if their union is connected. Since $\sigma(N_1)$ is connected, the union of the relevant boxes is connected.

By Theorem 2.1 we can approximate N_1 and N_2 within ϵ by normal operators M_1 and M_2 in \mathfrak{A} with finite spectrum. By Lemma 2.2, by the semicontinuity of the spectrum, and by perturbing eigenvalues, we can assume that $\sigma(M_q)$ is precisely the centres of the relevant boxes and $\|N_q - M_q\| \leq 3\epsilon$ for all $q \in \{1, 2\}$.

We claim that there exists a unitary $U \in \mathfrak{A}$ such that $\|M_1 - U^* M_2 U\| \leq 2\epsilon$. Consider a tree \mathcal{T} in \mathbb{C} whose vertices are the centres of the relevant boxes and whose edges are straight lines that connect vertices in adjacent relevant boxes. Consider a leaf of \mathcal{T} . We can identify this leaf with the spectral projections of M_1 and M_2 corresponding to the eigenvalue defined by the vertex. We can then apply the ‘back and forth’ technique illustrated above to embed the spectral projection of M_1 under the corresponding spectral projection of M_2 and the remaining spectral projection of M_2 under a spectral projection of M_1 corresponding to the adjacent vertex of the leaf (which is within 2ϵ). By considering \mathcal{T} with the above leaf removed, we then have a smaller tree. By continually repeating this ‘back and forth’-crossing technique, we are eventually left with the trivial tree. As before, K -theory implies the remaining projections are Murray-von Neumann equivalent. It is then possible to use the partial isometries from the ‘back and forth’ construction to create a unitary with the desired properties. \square

Our next goal is to remove the condition ‘ $\sigma(N_1)$ is connected’ from Lemma 2.11. Unfortunately, two normal operators having equal spectrum is not enough to guarantee that the normal operators are approximately unitarily equivalent (even in the case that $K_1(\mathfrak{A})$ is trivial). The technicality is the same as why two projections

in $\mathcal{B}(\mathcal{H})$ are not always approximately unitarily equivalent. To see this, we note the following lemmas.

Lemma 2.12. *Let \mathfrak{A} be a unital C^* -algebra and let $P, Q \in \mathfrak{A}$ be projections. If there exists an element $V \in \mathfrak{A}^{-1}$ such that*

$$\|Q - V P V^{-1}\| < \frac{1}{2}$$

then P and Q are Murray-von Neumann equivalent.

Proof. Let $P_0 := V P V^{-1} \in \mathfrak{A}$ and let $Z := P_0 Q + (I_{\mathfrak{A}} - P_0)(I_{\mathfrak{A}} - Q) \in \mathfrak{A}$. Hence P_0 is an idempotent and it is clear that

$$\begin{aligned} \|Z - I_{\mathfrak{A}}\| &= \|(P_0 Q + (I_{\mathfrak{A}} - P_0)(I_{\mathfrak{A}} - Q)) - (Q + (I_{\mathfrak{A}} - Q))\| \\ &\leq \|(P_0 - I_{\mathfrak{A}})Q\| + \|((I_{\mathfrak{A}} - P_0) - I_{\mathfrak{A}})(I_{\mathfrak{A}} - Q)\| \\ &= \|(P_0 - Q)Q\| + \|((I_{\mathfrak{A}} - P_0) - (I_{\mathfrak{A}} - Q))(I_{\mathfrak{A}} - Q)\| \\ &\leq \|P_0 - Q\| + \|Q - P_0\| < 1. \end{aligned}$$

Hence $Z \in \mathfrak{A}^{-1}$. Therefore, if U is the partial isometry in the polar decomposition of Z , $Z = U|Z|$ and U is a unitary element of \mathfrak{A} .

We claim that $U Q U^* = P_0$. To see this, we notice that $U = Z|Z|^{-1}$, $ZQ = P_0 Q = P_0 Z$, and

$$Z^* Z = Q P_0 Q + (I_{\mathfrak{A}} - Q)(I_{\mathfrak{A}} - P_0)(I_{\mathfrak{A}} - Q).$$

Thus $Q Z^* Z = Q P_0 Q = Z^* Z Q$ so Q commutes with $Z^* Z$. Hence Q commutes with $C^*(Z^* Z)$ and thus Q commutes with $|Z|^{-1}$. Thus

$$\begin{aligned} U Q U^* &= Z|Z|^{-1} Q |Z|^{-1} Z^* \\ &= Z Q |Z|^{-2} Z^* \\ &= P_0 Z |Z|^{-2} Z^* = P_0 \end{aligned}$$

as claimed.

Therefore $Q = (U^* V) P (U^* V)^{-1}$ where $U^* V \in \mathfrak{A}^{-1}$. It is standard to verify that if W is the partial isometry in the polar decomposition of $U^* V$ then W is a unitary such that $Q = W P W^*$ (see [RLL, Proposition 2.2.5]). Therefore $P \sim_u Q$ and thus P and Q are Murray-von Neumann equivalent. \square

Lemma 2.13. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra and let P and Q be projections in \mathfrak{A} . Then $P \sim_u Q$ if and only if $P \sim_{au} Q$ if and only if $Q \in \overline{\mathcal{S}(P)}$ only if P and Q are Murray-von Neumann equivalent. If $P \neq I_{\mathfrak{A}}$ and $Q \neq I_{\mathfrak{A}}$, then $P \sim_u Q$ whenever P and Q are Murray-von Neumann equivalent.*

Proof. Clearly $P \sim_u Q$ implies $P \sim_{au} Q$ and $P \sim_{au} Q$ implies $Q \in \overline{\mathcal{S}(P)}$. Moreover $Q \in \overline{\mathcal{S}(P)}$ implies P and Q are Murray-von Neumann equivalent by Lemma 2.12.

Suppose $P \neq I_{\mathfrak{A}}$ and $Q \neq I_{\mathfrak{A}}$ are Murray-von Neumann equivalent. Clearly the result holds if $P = 0$ or $Q = 0$ so suppose otherwise. Thus

$$[I_{\mathfrak{A}} - P]_0 = [I_{\mathfrak{A}}]_0 - [P]_0 = [I_{\mathfrak{A}}]_0 - [Q]_0 = [I_{\mathfrak{A}} - Q]_0.$$

Since \mathfrak{A} is a unital, simple, purely infinite C^* -algebra, the above implies $I_{\mathfrak{A}} - P$ and $I_{\mathfrak{A}} - Q$ are Murray-von Neumann equivalent. If V and W are partial isometries in \mathfrak{A} such that $V^* V = P$, $V V^* = Q$, $W^* W = I_{\mathfrak{A}} - P$, and $W W^* = I_{\mathfrak{A}} - Q$ then $U := V + W \in \mathfrak{A}$ is a unitary operator such that $P = U^* Q U$. Hence $P \sim_u Q$. \square

The above shows that if \mathfrak{A} is a unital, simple, purely infinite C*-algebra with $K_0(\mathfrak{A})$ being non-trivial, there exists two projections $P, Q \in \mathfrak{A}$ with $\sigma(P) = \sigma(Q) = \{0, 1\}$ that are not approximately unitarily equivalent. Thus knowledge of the spectrum is not enough to complete our classification.

To avoid the above technicality, we will describe an additional condition for two normal operators to be approximately unitarily equivalent in a unital, simple, purely infinite C*-algebra. The construction of this conditions makes use of the analytical functional calculus.

Lemma 2.14. *Let \mathfrak{A} be a unital C*-algebra, let $A, B \in \mathfrak{A}$, and let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a function that is analytic on an open neighbourhood U of $\sigma(A) \cup \sigma(B)$. If $A \in \overline{\mathcal{S}(B)}$ then $f(A) \in \overline{\mathcal{S}(f(B))}$. Similarly if $A \sim_{au} B$ then $f(A) \sim_{au} f(B)$.*

Proof. Let $(V_n)_{n \geq 1}$ be a sequence of invertible elements in \mathfrak{A} such that

$$\lim_{n \rightarrow \infty} \|A - V_n B V_n^{-1}\| = 0.$$

Let γ be any compact, rectifiable curve inside U such that $(\sigma(A) \cup \sigma(B)) \cap \gamma = \emptyset$, $Ind_\gamma(z) \in \{0, 1\}$ for all $z \in \mathbb{C} \setminus \gamma$, $Ind_\gamma(z) = 1$ for all $z \in \sigma(A) \cup \sigma(B)$, and $\{z \in \mathbb{C} \mid Ind_\gamma(z) \neq 0\} \subseteq U$. Then

$$\begin{aligned} & f(A) - V_n f(B) V_n^{-1} \\ &= \frac{1}{2\pi i} \int_\gamma f(z) ((zI_{\mathfrak{A}} - A)^{-1} - V_n(zI_{\mathfrak{A}} - B)^{-1} V_n^{-1}) dz \\ &= \frac{1}{2\pi i} \int_\gamma f(z) ((zI_{\mathfrak{A}} - A)^{-1} - (zI_{\mathfrak{A}} - V_n B V_n^{-1})^{-1}) dz \\ &= \frac{1}{2\pi i} \int_\gamma f(z) (zI_{\mathfrak{A}} - A)^{-1} (A - V_n B V_n^{-1}) (zI_{\mathfrak{A}} - V_n B V_n^{-1})^{-1} dz. \end{aligned}$$

Hence $\|f(A) - V_n f(B) V_n^{-1}\|$ is at most

$$\frac{length(\gamma) \|A - V_n B V_n^{-1}\|}{2\pi} \sup_{z \in \gamma} |f(z)| \| (zI_{\mathfrak{A}} - A)^{-1} \| \| (zI_{\mathfrak{A}} - V_n B V_n^{-1})^{-1} \|.$$

Provided $\|A - V_n B V_n^{-1}\| \| (zI_{\mathfrak{A}} - A)^{-1} \| < 1$ for all $z \in \gamma$, the second resolvent equation can be used to show that

$$\| (zI_{\mathfrak{A}} - V_n B V_n^{-1})^{-1} \| \leq \frac{\| (zI_{\mathfrak{A}} - A)^{-1} \|}{1 - \|A - V_n B V_n^{-1}\| \| (zI_{\mathfrak{A}} - A)^{-1} \|}$$

for all $z \in \gamma$. Since $\lim_{n \rightarrow \infty} \|A - V_n B V_n^{-1}\| = 0$, γ is compact, and the resolvent function of an operator is continuous on the resolvent, $\|f(A) - V_n f(B) V_n^{-1}\|$ is at most

$$\frac{length(\gamma) \|A - V_n B V_n^{-1}\|}{2\pi} \sup_{z \in \gamma} |f(z)| \frac{\| (zI_{\mathfrak{A}} - A)^{-1} \|^2}{1 - \|A - V_n B V_n^{-1}\| \| (zI_{\mathfrak{A}} - A)^{-1} \|}$$

for sufficiently large n . Since the resolvent function is a continuous function on the resolvent of an operator and γ is compact, the above supremum is finite and tends to

$$\sup_{z \in \gamma} |f(z)| \| (zI_{\mathfrak{A}} - A)^{-1} \|^2$$

as $n \rightarrow \infty$. Thus, as

$$\lim_{n \rightarrow \infty} \|A - V_n B V_n^{-1}\| = 0$$

and $length(\gamma)$ is finite, $f(A) \in \overline{\mathcal{S}(f(B))}$.

The proof that $A \sim_{au} B$ implies $f(A) \sim_{au} f(B)$ follows directly by replacing the invertible elements V_n with unitary operators. \square

Finally, with the above and the arguments used in Lemma 2.11, we can state and prove our main result of the section.

Theorem 2.15. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators. Suppose*

- (1) $\sigma(N_1) = \sigma(N_2)$,
- (2) $\lambda I_{\mathfrak{A}} - N_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1, 2\}$, and
- (3) for every function $f : \mathbb{C} \rightarrow \mathbb{C}$ that is analytic on an open neighbourhood U of $\sigma(N_1)$ with $f(U) \subseteq \{0, 1\}$, the projections $f(N_1)$ and $f(N_2)$ are Murray-von Neumann equivalent.

Then $N_1 \sim_{au} N_2$.

Proof. Fix $\epsilon > 0$ and consider the ϵ -grid used in Lemma 2.11. We label the box $B_{n,m}$ relevant if $B_{n,m} \cap \sigma(N_1) \neq \emptyset$. Let K be the union of the relevant boxes. Since $\sigma(N_1)$ is compact, K has finitely many connected components. Let L_1, \dots, L_k be the connected components of K . By construction $\text{dist}(L_i, L_j) \geq \epsilon$ for all $i \neq j$. Therefore, if f_i is the characteristic function of L_i , the third assumptions of the theorem implies $f_i(N_1)$ and $f_i(N_2)$ are Murray-von Neumann equivalent for each $i \in \{1, \dots, k\}$.

Note the second assumption of the theorem implies that there exists normal operators M_1 and M_2 in \mathfrak{A} with finite spectrum such that $\|N_q - M_q\| < \epsilon$ for all $q \in \{1, 2\}$. By an application of Lemma 2.2, by the semicontinuity of the spectrum, and by small perturbations, we can assume that M_q has spectrum contained in K and $\sigma(M_q) \cap B_{n,m} \neq \emptyset$ for all relevant boxes $B_{n,m}$ and $q \in \{1, 2\}$. Furthermore, since each f_i extends to a continuous function on an open neighbourhood of K , we can assume that $\|f_i(N_q) - f_i(M_q)\| < \frac{1}{2}$ for all $i \in \{1, \dots, k\}$ and $q \in \{1, 2\}$ by properties of the continuous functional calculus. Therefore, for each $i \in \{1, \dots, k\}$ and $q \in \{1, 2\}$, $f_i(N_q)$ and $f_i(M_q)$ can be assumed to be Murray-von Neumann equivalent by Lemma 2.12. Since $f_i(N_1)$ and $f_i(N_2)$ are Murray-von Neumann equivalent for each $i \in \{1, \dots, k\}$, $f_i(M_1)$ and $f_i(M_2)$ are Murray-von Neumann equivalent for each $i \in \{1, \dots, k\}$. By perturbing the spectrum of M_1 and M_2 inside each L_i , we can assume that $\sigma(M_q)$ is precisely the centres of the relevant boxes for all $q \in \{1, 2\}$, $f_i(M_1)$ and $f_i(M_2)$ are Murray-von Neumann equivalent for each $i \in \{1, \dots, k\}$, and $\|N_q - M_q\| < 3\epsilon$ for all $q \in \{1, 2\}$.

Next we apply the ‘back and forth’ argument of Lemma 2.11 to the spectrum of M_1 and M_2 in each L_i separately. This process can be applied to each L_i separately as in Lemma 2.11 due to the fact that $f_i(M_1)$ and $f_i(M_2)$ are Murray-von Neumann equivalent so the final step of the construction (that is, $R_n^{(1)}$ and $Q_n^{(2)}$ are Murray-von Neumann equivalent) can be completed. Thus, for each $i \in \{1, \dots, k\}$, the ‘back and forth’ process produces a partial isometry $V_i \in \mathfrak{A}$ such that $V_i^* V_i = f_i(M_1)$, $V_i V_i^* = f_i(M_2)$, and $\|M_1 f_i(M_1) - V_i^* M_2 f_i(M_2) V_i\| \leq 2\epsilon$. Therefore, if $U := \sum_{i=1}^k V_i$ then $U \in \mathfrak{A}$ is a unitary as

$$\sum_{i=1}^k f_i(M_1) = I_{\mathfrak{A}} = \sum_{i=1}^k f_i(M_2)$$

are sums of orthogonal projections. Moreover, a trivial computation shows

$$\|M_1 - U^* M_2 U\| \leq 2\epsilon$$

so

$$\|N_1 - U^*N_2U\| \leq 8\epsilon$$

completing the proof. \square

Corollary 2.16. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra such that $K_1(\mathfrak{A})$ is trivial and let $N_1, N_2 \in \mathfrak{A}$ be normal operators. Then $N_1 \sim_{au} N_2$ if and only if*

- (1) $\sigma(N_1) = \sigma(N_2)$, and
- (2) *for every function $f : \mathbb{C} \rightarrow \mathbb{C}$ that is analytic on an open neighbourhood U of $\sigma(N_1)$ with $f(U) \subseteq \{0, 1\}$, the projections $f(N_1)$ and $f(N_2)$ are Murray-von Neumann equivalent.*

Proof. One direction follows from Theorem 2.15 and the fact that $K_1(\mathfrak{A})$ is trivial implies $\mathfrak{A}^{-1} = \mathfrak{A}_0^{-1}$ by [Cu, Theorem 1.9]. The other direction follows from Lemma 2.14 and Lemma 2.13. \square

The following is the obvious question raised by Proposition 2.10, Lemma 2.14, and Theorem 2.15.

Question 2.17. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators. Is $N_1 \sim_{au} N_2$ if and only if*

- (1) $\sigma(N_1) = \sigma(N_2)$,
- (2) $\lambda I_{\mathfrak{A}} - N_1$ and $\lambda I_{\mathfrak{A}} - N_2$ are in the same connected component of \mathfrak{A}^{-1} for all $\lambda \notin \sigma(N_1)$, and
- (3) *for every function $f : \mathbb{C} \rightarrow \mathbb{C}$ that is analytic on an open neighbourhood U of $\sigma(N_1)$ with $f(U) \subseteq \{0, 1\}$, the projections $f(N_1)$ and $f(N_2)$ are Murray-von Neumann equivalent?*

To conclude this section, we note that Corollary 2.4, Corollary 2.16, and Theorem 2.15 respectively imply the following results that are similar to [KP, Theorem 1.13].

Corollary 2.18. *Let \mathfrak{B} be a unital C^* -algebra generated by a single normal operator. Any two injective unital $*$ -homomorphisms from \mathfrak{B} to \mathcal{O}_2 are approximately unitarily equivalent.*

Corollary 2.19. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra and let \mathfrak{B} be a unital C^* -algebra generated by a single normal operator whose spectrum is connected. If $K_1(\mathfrak{A})$ is trivial, any two injective unital $*$ -homomorphisms from \mathfrak{B} to \mathfrak{A} are approximately unitarily equivalent.*

Corollary 2.20. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra and let \mathfrak{B} be a unital C^* -algebra generated by a single self-adjoint operator whose spectrum is connected. Any two injective unital $*$ -homomorphisms from \mathfrak{B} to \mathfrak{A} are approximately unitarily equivalent.*

Note that discussions presented in this chapter show that neither ‘ $K_1(\mathfrak{A})$ is trivial’ nor ‘whose spectrum is connected’ can be removed from the suppositions of Corollary 2.19.

3. DISTANCE BETWEEN UNITARY ORBITS OF NORMAL OPERATORS

Significant progress has been made in determining the distance between two unitary orbits of bounded operators on a complex, infinite dimensional Hilbert space (see [Da2] and [Da3]). In terms of determining the distance between unitary orbits of operators inside other C^* -algebras, [Da1] makes significant progress for the Calkin algebra and [HN] makes significant progress for semifinite factors.

In this section we will provide some bounds for the distance between the unitary orbits of two normal operator in unital, simple, purely infinite C^* -algebras. In particular, Corollary 3.8 can be used to deduce Theorem 2.15. We begin with the following definition that is common in the discussion of the distance between unitary orbits.

Definition 3.1. Let X and Y be subsets of \mathbb{C} . The Hausdorff distance between X and Y , denoted $d_H(X, Y)$, is

$$d_H(X, Y) := \max \left\{ \sup_{x \in X} \text{dist}(x, Y), \sup_{y \in Y} \text{dist}(y, X) \right\}.$$

To begin, we note the following well-known lower bound.

Proposition 3.2 ([Da2, Proposition 2.1]). *Let \mathfrak{A} be a unital C^* -algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators. Then*

$$\text{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \geq d_H(\sigma(N_1), \sigma(N_2)).$$

For our discussions of the distance between unitary orbits of normal operators in unital, simple, purely infinite C^* -algebras, we first turn our attention to the Cuntz algebra \mathcal{O}_2 . As $K_0(\mathcal{O}_2)$ and $K_1(\mathcal{O}_2)$ are trivial, we are led to the following generalization of [HN, Theorem 1.5] whose proof is identical to the one given below.

Proposition 3.3 (see [HN, Theorem 1.5]). *Let \mathfrak{A} be a unital C^* -algebra such that \mathfrak{A} has property weak (FN), any two non-zero projections in \mathfrak{A} are Murray-von Neumann equivalent, and every non-zero projection in \mathfrak{A} is properly infinite. Let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that $\lambda I_{\mathfrak{A}} - N_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1, 2\}$. Then*

$$\text{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) = d_H(\sigma(N_1), \sigma(N_2)).$$

Proof. One inequality follows from Proposition 3.2. Let $\epsilon > 0$. Since \mathfrak{A} has weak (FN), the conditions on N_1 and N_2 imply that there exists two normal operators $M_1, M_2 \in \mathfrak{A}$ with finite spectrum such that $\|N_q - M_q\| < \epsilon$ for all $q \in \{1, 2\}$. By Lemma 2.2, by the semicontinuity of the spectrum, and by applying small perturbations, we may assume that $\sigma(M_q) \subseteq \sigma(N_q)$ and $\sigma(M_q)$ is an ϵ -net for $\sigma(N_q)$ for all $q \in \{1, 2\}$.

Let X be the set of all ordered pairs $(\lambda, \mu) \in \sigma(M_1) \times \sigma(M_2)$ such that either

$$|\lambda - \mu| = \text{dist}(\lambda, \sigma(M_2)) \text{ or } |\lambda - \mu| = \text{dist}(\mu, \sigma(M_1)).$$

For each $\lambda \in \sigma(M_1)$ and $\mu \in \sigma(M_2)$, let $n_\lambda := |\{(\lambda, \zeta) \in X\}|$ and $m_\mu := |\{(\zeta, \mu) \in X\}|$. Clearly $n_\lambda \geq 1$ for all $\lambda \in \sigma(M_1)$, $m_\mu \geq 1$ for all $\mu \in \sigma(M_2)$, and $\sum_{\lambda \in \sigma(M_1)} n_\lambda = \sum_{\mu \in \sigma(M_2)} m_\mu$.

Since every projection in \mathfrak{A} is properly infinite, we can write

$$M_1 = \sum_{\lambda \in \sigma(M_1)} \sum_{k=1}^{n_\lambda} \lambda P_{\lambda,k} \quad \text{and} \quad M_2 = \sum_{\mu \in \sigma(M_2)} \sum_{k=1}^{m_\mu} \mu Q_{\mu,k}$$

where $\{\{P_{\lambda,k}\}_{k=1}^{n_\lambda}\}_{\lambda \in \sigma(M_1)}$ and $\{\{Q_{\mu,k}\}_{k=1}^{m_\mu}\}_{\mu \in \sigma(M_2)}$ are sets of non-zero orthogonal projections in \mathfrak{A} each of which sums to the identity. Since all projections in \mathfrak{A} are Murray-von Neumann equivalent, using X we can pair off the projections in these finite sums to obtain a unitary $U \in \mathfrak{A}$ such that

$$\|M_1 - UM_2U^*\| \leq \sup\{|\lambda - \mu| \mid (\lambda, \mu) \in X\} = d_H(\sigma(M_1), \sigma(M_2)).$$

Hence

$$\text{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \leq 2\epsilon + d_H(\sigma(M_1), \sigma(M_2)).$$

Since $\sigma(M_1)$ is an ϵ -net for $\sigma(N_1)$, and $\sigma(M_2)$ is an ϵ -net for $\sigma(N_2)$,

$$d_H(\sigma(M_1), \sigma(M_2)) \leq d_H(\sigma(N_1), \sigma(N_2)) + \epsilon$$

completing the proof. \square

Unfortunately Proposition 3.3 does not completely generalize to unital, simple, purely infinite C^* -algebras with non-trivial K_0 -group. However, the following is a step in the right direction.

Lemma 3.4. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that $\lambda I_{\mathfrak{A}} - N_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1, 2\}$. If $\sigma(N_1)$ is connected then*

$$\text{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) = d_H(\sigma(N_1), \sigma(N_2)).$$

Proof. One inequality follows from Proposition 3.2. The proof of the other inequality is a more complicated ‘back and forth’ argument. Fix $\epsilon > 0$ and let $B_{n,m}$ be as in Lemma 2.11. For each $q \in \{1, 2\}$, we will say that $B_{n,m}$ is N_q -relevant if $B_{n,m} \cap \sigma(N_q) \neq \emptyset$. By Theorem 2.1 there exists normal operators $M_1, M_2 \in \mathfrak{A}$ with finite spectrum such that $\|N_q - M_q\| < \epsilon$ for all $q = \{1, 2\}$. By Lemma 2.2, by the semicontinuity of the spectrum, and by a small perturbation, we can assume that $\sigma(M_q)$ is precisely the centres of the N_q -relevant boxes and $\|N_q - M_q\| \leq 3\epsilon$. For each $q \in \{1, 2\}$ and $\lambda \in \sigma(M_q)$, let $P_\lambda^{(q)}$ be the non-zero spectral projection of M_q corresponding to λ .

To begin our ‘back and forth’ argument, we will construct a bipartite graph, \mathcal{G} , using $\sigma(M_1)$ and $\sigma(M_2)$ as vertices (where we have two vertices for λ if $\lambda \in \sigma(M_1) \cap \sigma(M_2)$). The process for constructing the edges in \mathcal{G} is as follows: for each $i, j \in \{1, 2\}$ with $i \neq j$ and each $\lambda \in \sigma(M_i)$, for every $\mu \in \sigma(M_j)$ such that

$$|\lambda - \mu| \leq 2\epsilon + d_H(\sigma(N_1), \sigma(N_2))$$

(note that at least one such μ exists) add edges to \mathcal{G} from μ to λ and the centre of any N_i -relevant box adjacent (including diagonally adjacent) to the N_i -relevant box λ describes.

Clearly \mathcal{G} is a bipartite graph. Moreover, by construction, if $\lambda \in \sigma(M_1)$ and $\mu \in \sigma(M_2)$ are connected by an edge of \mathcal{G} then $|\lambda - \mu| \leq 4\epsilon + d_H(\sigma(N_1), \sigma(N_2))$. Finally, we claim that \mathcal{G} is connected. To see this, we note that since \mathcal{G} is bipartite and every vertex is the endpoint of at least one edge, it suffices to show for each pair $\lambda, \mu \in \sigma(M_1)$ that there exists a path from λ to μ . Fix a pair $\lambda, \mu \in \sigma(M_1)$. Since $\sigma(N_1)$ is connected, the union of the N_1 -relevant boxes is connected so there exists a finite sequence $\lambda = \lambda_0, \lambda_1, \dots, \lambda_k = \mu$ where $\lambda_{\ell-1}$ and λ_ℓ are centres of adjacent N_1 -relevant boxes for all $\ell \in \{1, \dots, k\}$. However $\lambda_{\ell-1}$ and λ_ℓ are connected in \mathcal{G} (via an element of $\sigma(M_2)$) by construction. Hence the claim follows.

Now that \mathcal{G} is constructed, we will progressively remove vertices and edges from \mathcal{G} and modify the non-zero projections $\left\{ \left\{ P_\lambda^{(q)} \right\}_{\lambda \in \sigma(M_j)} \right\}_{q \in \{1,2\}}$ in a specific manner to construct partial isometries in \mathfrak{A} that will enable us to create a unitary $U \in \mathfrak{A}$ such that

$$\|M_1 - U^* M_2 U\| \leq 4\epsilon + d_H(\sigma(N_1), \sigma(N_2)).$$

Since \mathcal{G} is a connected graph, there exists a $j \in \{1, 2\}$ and a vertex $\lambda \in \sigma(M_j)$ in \mathcal{G} whose removal (along with all edges connecting that vertex) does not disconnect \mathcal{G} . Choose any vertex μ in \mathcal{G} connected to λ by an edge. By the construction of \mathcal{G} $|\lambda - \mu| \leq 4\epsilon + d_H(\sigma(N_1), \sigma(N_2))$ and $\mu \in \sigma(M_i)$ where $i \in \{1, 2\} \setminus \{j\}$. Since \mathfrak{A} is a unital, simple, purely infinite C^* -algebra and $P_\mu^{(i)}$ is non-zero, there exists non-zero projections $Q_\mu^{(i)}$ and $R_\mu^{(i)}$ in \mathfrak{A} such that $P_\lambda^{(j)}$ and $Q_\mu^{(i)}$ are Murray-von Neumann equivalent and $P_\mu^{(i)} = Q_\mu^{(i)} + R_\mu^{(i)}$. To complete our recursive step, remove λ from \mathcal{G} (so \mathcal{G} will still be a connected, bipartite graph), remove $P_\lambda^{(j)}$ from our list of projections, and replace $P_\mu^{(i)}$ with $R_\mu^{(i)}$ in our list of projections.

Continue the recursive process in the above paragraph until two vertices are left in \mathcal{G} . By the construction of \mathcal{G} , one of these two remaining vertices is a non-zero subprojection of a spectral projection of M_1 and the other is a non-zero subprojection of a spectral projection of M_2 . These two projections are Murray-von Neumann equivalent by the same K -theory argument used in Lemma 2.11.

By the same arguments as Lemma 2.11, the Murray-von Neumann equivalence of the projections created in the above process allows us to create partial isometries and thus, by taking a sum, a unitary $U \in \mathfrak{A}$ with the claimed property. Hence

$$\|N_1 - U^* N_2 U\| \leq 10\epsilon + d_H(\sigma(N_1), \sigma(N_2)).$$

As $\epsilon > 0$ was arbitrary, the result follows. \square

The above proof can be modified to show the following results.

Corollary 3.5. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that $\lambda I_{\mathfrak{A}} - N_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1, 2\}$. Suppose for each $q \in \{1, 2\}$ that $\sigma(N_q) = \bigcup_{i=1}^n K_i^{(q)}$ is a disjoint union of connected, compact sets. Let $\chi_i^{(q)}$ be the characteristic function of $K_i^{(q)}$ for all $q \in \{1, 2\}$ and $i \in \{1, \dots, n\}$. If $\chi_i^{(1)}(N_1)$ and $\chi_i^{(2)}(N_2)$ are Murray-von Neumann equivalent for all $i \in \{1, \dots, n\}$ then*

$$\text{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \leq \max_{i \in \{1, \dots, n\}} d_H(K_i^{(1)}, K_i^{(2)}).$$

Proof. Fix $\epsilon > 0$. The condition that ' $\chi_i^{(1)}(N_1)$ and $\chi_i^{(2)}(N_2)$ are Murray-von Neumann equivalent' allows the arguments of Lemma 3.4 to be applied on each component to produce a partial isometry $V_i \in \mathfrak{A}$ such that $V_i^* V_i = \chi_i^{(1)}(N_1)$, $V_i V_i^* = \chi_i^{(2)}(N_2)$, and

$$\|N_1 \chi_i^{(1)}(N_1) - V_i^* N_2 \chi_i^{(2)}(N_2) V_i\| < \epsilon + d_H(K_i^{(1)}, K_i^{(2)}).$$

If $U := \sum_{i=1}^n V_i \in \mathfrak{A}$ then U is a unitary operator such that

$$\|N_1 - U^* N_2 U\| < \epsilon + \max_{i \in \{1, \dots, n\}} d_H(K_i^{(1)}, K_i^{(2)}).$$

Hence the result follows. \square

Lemma 3.6. *Let \mathfrak{A} be a unital, simple, purely infinite C*-algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that $\lambda I_{\mathfrak{A}} - N_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1, 2\}$. Suppose that $\sigma(N_1) \cup \sigma(N_2)$ is connected. Then*

$$\text{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) = d_H(\sigma(N_1), \sigma(N_2)).$$

Proof. The only caveat in the proof of Lemma 3.4 is that we require that the bipartite graph \mathcal{G} is connected. For each $q \in \{1, 2\}$ let $K^{(q)}$ be the union of the N_q -relevant $B_{n,m}$. Therefore, each $K^{(q)}$ is a union of finitely many connected components. For each $q \in \{1, 2\}$ let $\left\{K_k^{(q)}\right\}_{k=1}^{n_q}$ be the connected components of $K^{(q)}$. By the construction of \mathcal{G} in Lemma 3.4, for each $q \in \{1, 2\}$ all vertices from $\sigma(M_q)$ inside $K_k^{(q)}$ are connected in \mathcal{G} . Moreover, for $i, j \in \{1, 2\}$ with $i \neq j$, each vertex from $\sigma(M_j)$ inside $K_k^{(i)}$ is connected to each vertex of $K_\ell^{(j)}$ provided that $K_k^{(i)} \cup K_\ell^{(j)}$ is connected. Since $\sigma(N_1) \cup \sigma(N_2)$ is connected,

$$\left(\bigcup_{k=1}^{n_1} K_k^{(1)}\right) \cup \left(\bigcup_{k=1}^{n_2} K_k^{(2)}\right)$$

is connected and thus \mathcal{G} is connected. The remainder of the proof then follows as in Lemma 3.4. \square

Our final upper bound for the distance between two unitary orbits of normal operators in unital, simple, purely infinite C*-algebras is the following which combines the ideas of the above two results.

Proposition 3.7. *Let \mathfrak{A} be a unital, simple, purely infinite C*-algebra, let $\epsilon > 0$, and let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that $\lambda I_{\mathfrak{A}} - N_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1, 2\}$. Let $B_{n,m}$ be as in Lemma 2.11. For each $q \in \{1, 2\}$, we will say that $B_{n,m}$ is N_q -relevant if $B_{n,m} \cap \sigma(N_q) \neq \emptyset$. For each $q \in \{1, 2\}$ let $K^{(q)}$ be the union of the N_q -relevant boxes. Suppose there exists an $n \in \mathbb{N}$ such that for each $q \in \{1, 2\}$ we can write $K^{(q)} = \bigcup_{i=1}^n K_i^{(q)}$ where each $K_i^{(q)}$ is the union of finitely many connected sets, $K_i^{(1)} \cup K_i^{(2)}$ is connected, and*

$$\text{dist}\left(K_i^{(1)} \cup K_i^{(2)}, K_j^{(1)} \cup K_j^{(2)}\right) > 0$$

whenever $i \neq j$. Let $\chi_i^{(q)}$ be the characteristic function of $K_i^{(q)}$ for all $q \in \{1, 2\}$ and $i \in \{1, \dots, n\}$. If $\chi_i^{(1)}(N_1)$ and $\chi_i^{(2)}(N_2)$ are Murray-von Neumann equivalent for all $i \in \{1, \dots, n\}$ then

$$\text{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \leq 10\epsilon + \max_{i \in \{1, \dots, n\}} d_H\left(K_i^{(1)}, K_i^{(2)}\right).$$

Proof. Let \mathcal{G} be the bipartite graph described in Lemma 3.4 for this selection of ϵ . The graph \mathcal{G} is not connected but the conditions of this proposition allows the proof of Lemma 3.6 to be performed on the vertices of $K_i^{(1)} \cup K_i^{(2)}$ separately to construct partial isometries $V_i \in \mathfrak{A}$ such that $V_i^* V_i = \chi_i^{(1)}(N_1)$, $V_i V_i^* = \chi_i^{(2)}(N_2)$, and

$$\left\|N_1 \chi_i^{(1)}(N_1) - V_i^* N_2 \chi_i^{(2)}(N_2) V_i\right\| \leq 10\epsilon + d_H\left(K_i^{(1)}, K_i^{(2)}\right).$$

If $U := \sum_{i=1}^k V_i \in \mathfrak{A}$ then U is a unitary operator such that

$$\|N_1 - U^* N_2 U\| < 10\epsilon + \max_{i \in \{1, \dots, n\}} d_H(K_i^{(1)}, K_i^{(2)}).$$

Hence the result follows. \square

Corollary 3.8. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that $\lambda I_{\mathfrak{A}} - N_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1, 2\}$. Suppose*

- (1) $\sigma(N_2) \subseteq \sigma(N_1)$, and
- (2) *for every function $f : \mathbb{C} \rightarrow \mathbb{C}$ that is analytic on an open neighbourhood U of $\sigma(N_1)$ with $f(U) \subseteq \{0, 1\}$, the projections $f(N_1)$ and $f(N_2)$ are Murray-von Neumann equivalent.*

Then

$$\text{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) = d_H(\sigma(N_1), \sigma(N_2)).$$

Proof. One inequality follows from Proposition 3.2. Let $\epsilon > 0$. The two conditions listed in this corollary imply the suppositions of Proposition 3.7 are satisfied for this choice of ϵ . Hence

$$\text{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \leq 10\epsilon + d_H(\sigma(N_1), \sigma(N_2)).$$

As ϵ was arbitrary, the result follows. \square

We have made use of the equivalence of certain spectral projections in the creation of all of the above bounds. To illustrate the necessity of these assumptions, we note the following example.

Example 3.9. Let P and Q be non-trivial projections in \mathcal{O}_3 with $[P]_0 \neq [Q]_0$. Then $\sigma(P) = \sigma(Q)$ yet $\text{dist}(\mathcal{U}(P), \mathcal{U}(Q)) \geq 1$ or else P and Q would be Murray-von Neumann equivalent (see [RLL, Proposition 2.2.4] and [RLL, Proposition 2.2.7]).

In particular, we have the following quantitative version of the above example.

Proposition 3.10. *Let \mathfrak{A} be a unital C^* -algebra, let $N_1, N_2 \in \mathfrak{A}$ be normal operators, and let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a function that is analytic on an open neighbourhood U of $\sigma(N_1) \cup \sigma(N_2)$ with $f(U) \subseteq \{0, 1\}$. Let γ be a compact, rectifiable curve inside U with $(\sigma(N_1) \cup \sigma(N_2)) \cap \gamma = \emptyset$, $\text{Ind}_{\gamma}(z) \in \{0, 1\}$ for all $z \in \mathbb{C} \setminus \gamma$, $\text{Ind}_{\gamma}(z) = 1$ for all $z \in \sigma(N_1) \cup \sigma(N_2)$, and $\{z \in \mathbb{C} \mid \text{Ind}_{\gamma}(z) \neq 0\} \subseteq U$. If $f(N_1)$ and $f(N_2)$ are not Murray-von Neumann equivalent then*

$$\text{dist}(\mathcal{U}(N_1), \mathcal{U}(N_2)) \geq \frac{2\pi}{l_0(\gamma) \sup_{z \in \gamma} \|(zI_{\mathfrak{A}} - N_1)^{-1}\| \|(zI_{\mathfrak{A}} - N_2)^{-1}\|}$$

where $l_0(\gamma)$ is the length of γ in the regions where $f(z) = 1$.

Proof. By the proof of Lemma 2.14, we know that $\|f(N_1) - Uf(N_2)U^*\|$ is at most

$$\frac{l_0(\gamma) \|N_1 - UN_2U^*\|}{2\pi} \sup_{z \in \gamma} \|(zI_{\mathfrak{A}} - N_1)^{-1}\| \|(zI_{\mathfrak{A}} - N_2)^{-1}\|$$

for all unitaries U in \mathfrak{A} . Since $f(N_1)$ and $f(N_2)$ are not Murray-von Neumann equivalent, $f(N_1)$ and $Uf(N_2)U^*$ are not Murray-von Neumann equivalent so

$$1 \leq \|f(N_1) - Uf(N_2)U^*\|$$

by [RLL, Proposition 2.2.5] and [RLL, Proposition 2.2.7]. Hence the result follows. \square

All of the above upper bounds on the distance between unitary orbits required the operators to have resolvent functions taking ranges in the connected component of the identity. The following is a simple estimate on why this was necessary.

Proposition 3.11. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra and let $A, B \in \mathfrak{A}$. Suppose there exist a $\lambda \in \mathbb{C} \setminus (\sigma(A) \cup \sigma(B))$ such that $\lambda I_{\mathfrak{A}} - A$ and $\lambda I_{\mathfrak{A}} - B$ are not in the same connected component of the identity. Then*

$$\text{dist}(\mathcal{U}(A), \mathcal{U}(B)) \geq \max \left\{ \|(\lambda I_{\mathfrak{A}} - A)^{-1}\|^{-1}, \|(\lambda I_{\mathfrak{A}} - B)^{-1}\|^{-1} \right\}.$$

Proof. Let $U \in \mathfrak{A}$ be a unitary operator. Since \mathfrak{A} is a unital, simple, purely infinite C^* -algebra, K-theory along with [Cu, Theorem 1.9] imply that $\lambda I_{\mathfrak{A}} - A$ and $U(\lambda I_{\mathfrak{A}} - B)U^*$ are not in the same connected component of the identity (see the proof of Proposition 2.10). Therefore

$$\begin{aligned} 1 &\leq \|I_{\mathfrak{A}} - (\lambda I_{\mathfrak{A}} - A)^{-1}U(\lambda I_{\mathfrak{A}} - B)U^*\| \\ &\leq \|(\lambda I_{\mathfrak{A}} - A)^{-1}\| \|(\lambda I_{\mathfrak{A}} - A) - U(\lambda I_{\mathfrak{A}} - B)U^*\| \\ &= \|(\lambda I_{\mathfrak{A}} - A)^{-1}\| \|A - UBU^*\| \end{aligned}$$

so

$$\text{dist}(\mathcal{U}(A), \mathcal{U}(B)) \geq \|(\lambda I_{\mathfrak{A}} - A)^{-1}\|^{-1}.$$

The result then follows by symmetry. \square

4. CLOSED SIMILARITY ORBITS OF NORMAL OPERATORS

The main goal of this section is to prove the following theorems that completely determine when one normal operator is in the closed similarity orbit of another normal operator in \mathcal{O}_n (with n finite) and in type III factors with separable predual. The two main results are similar in proof but pose slight technical differences and thus are listed separately.

Theorem 4.1. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra such that there exists an $n \in \mathbb{N}$ with $n[P]_0 = 0$ for all non-zero projections $P \in \mathfrak{A}$ (for example \mathcal{O}_{n+1} by [Cu, Theorem 3.7]). Let $N, M \in \mathfrak{A}$ be normal operators such that $\lambda I_{\mathfrak{A}} - M \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$ and let $\{K_\lambda\}_\Lambda$ be the connected components of $\sigma(N)$. Then $N \in \mathcal{S}(M)$ if and only if*

- (1) $\sigma(M) \subseteq \sigma(N)$,
- (2) $\sigma(M) \cap K_\lambda \neq \emptyset$ for all $\lambda \in \Lambda$,
- (3) $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$,
- (4) $\sigma(M) \cap K_\lambda$ contains a cluster point of $\sigma(M)$ whenever K_λ is not a singleton, and
- (5) for every function $f : \mathbb{C} \rightarrow \mathbb{C}$ that is analytic on an open neighbourhood U of $\sigma(N)$ with $f(U) \subseteq \{0, 1\}$, the projections $f(N)$ and $f(M)$ are Murray-von Neumann equivalent.

Theorem 4.2. *Let \mathfrak{A} be a unital C^* -algebra with the following properties;*

- (1) \mathfrak{A} has property weak (FN),
- (2) every non-zero projection in \mathfrak{A} is properly infinite, and
- (3) any two non-zero projections in \mathfrak{A} are Murray-von Neumann equivalent.

(For example, \mathcal{O}_2 and every type III factor with separable predual.)

Let $N, M \in \mathfrak{A}$ be normal operators such that $\lambda I_{\mathfrak{A}} - M \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$ and let $\{K_\lambda\}_\Lambda$ be the connected components of $\sigma(N)$. Then $N \in \mathcal{S}(M)$ if and only if

- (1) $\sigma(M) \subseteq \sigma(N)$,
- (2) $\sigma(M) \cap K_\lambda \neq \emptyset$ for all $\lambda \in \Lambda$,
- (3) $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$, and
- (4) $\sigma(M) \cap K_\lambda$ contains a cluster point of $\sigma(M)$ whenever K_λ is not a singleton.

Remarks 4.3. Note if $N \in \overline{\mathcal{S}(M)}$ then the first two conditions must hold by discussions from the introduction and the third condition follows from Proposition 2.10. The fifth condition of Theorem 4.1 is necessary by Lemma 2.14 and Lemma 2.12.

To see that the fourth conclusion is necessary, we note that if K_λ is not isolated in $\sigma(N)$ (that is, every open neighbourhood of K_λ intersects a different connected component of $\sigma(N)$) then the first two conditions imply that $\sigma(M) \cap K_\lambda$ contains a cluster point of $\sigma(M)$. Otherwise if K_λ is isolated in $\sigma(N)$, the characteristic function χ_{K_λ} of K_λ can be extended to an analytic function on a neighbourhood of $\sigma(N)$. Thus Lemma 2.14 implies $\chi_{K_\lambda}(N) \in \overline{\mathcal{S}(\chi_{K_\lambda}(M))}$. If $\sigma(M) \cap K_\lambda$ does not contain a cluster point of $\sigma(M)$ then $\chi_{K_\lambda}(M)$ must have finite spectrum. Hence there exists a non-zero polynomial p such that $p(\chi_{K_\lambda}(M)) = 0$. Clearly this implies $p(T) = 0$ for all $T \in \overline{\mathcal{S}(\chi_{K_\lambda}(M))}$ so $p(\chi_{K_\lambda}(N)) = 0$. Since K_λ is a connected, compact subset of $\sigma(N)$ that is not a singleton, this is impossible. Hence the third condition is necessary.

We note Remarks 4.3 show that conditions (1)-(4) are necessary for any unital, simple, purely infinite C^* -algebra and, in conjunction with Theorem 2.15, imply the following curious result.

Corollary 4.4. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra and let $N_1, N_2 \in \mathfrak{A}$ be normal operators such that $\lambda I_{\mathfrak{A}} - N_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1, 2\}$. If $N_1 \in \overline{\mathcal{S}(N_2)}$ and $N_2 \in \overline{\mathcal{S}(N_1)}$ then $N_1 \sim_{au} N_2$.*

To begin the proofs of Theorem 4.1 and Theorem 4.2, we note the following trivial result about similarity of operators in C^* -algebras.

Lemma 4.5. *Let \mathfrak{A} be a unital C^* -algebra, let $P \in \mathfrak{A}$ be a non-trivial projection, let $Z \in (I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$, and let $X \in \mathfrak{A}$ be such that $PX(I_{\mathfrak{A}} - P) = X$. If $\lambda \notin \sigma_{(I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)}(Z)$ then*

$$\lambda P + X + Z \sim \lambda P + Z.$$

Proof. Note that if $Y := X(\lambda(I_{\mathfrak{A}} - P) - Z)^{-1}$ then

$$T := I_{\mathfrak{A}} + Y$$

is invertible with

$$T^{-1} = I_{\mathfrak{A}} - Y.$$

A trivial computation shows

$$T(\lambda P + X + Z)T^{-1} = \lambda P + Z.$$

□

Corollary 4.6. *Let \mathfrak{A} be a unital C^* -algebra, let $n \in \mathbb{N}$, let $\lambda_1, \dots, \lambda_n$ be distinct complex scalars, let $\{P_j\}_{j=1}^n \subseteq \mathfrak{A}$ be a set of non-trivial orthogonal projections*

with $\sum_{j=1}^n P_j = I_{\mathfrak{A}}$, and let $\{A_{i,j}\}_{i,j=1}^n \subseteq \mathfrak{A}$ be such that $A_{i,j} = 0$ if $i \geq j$ and $P_i A_{i,j} P_j = A_{i,j}$ for all $i < j$. Then

$$\sum_{j=1}^n \lambda_j P_j + \sum_{i,j=1}^n A_{i,j} \sim \sum_{j=1}^n \lambda_j P_j.$$

Proof. By applying Lemma 4.5 with $P := P_1$, $Z := \sum_{j=1}^n \lambda_j P_j + \sum_{i,j=2}^n A_{i,j}$ (it is elementary to show that $\sigma_{(I_{\mathfrak{A}}-P)\mathfrak{A}(I_{\mathfrak{A}}-P)}(Z) = \{\lambda_2, \dots, \lambda_n\}$ so $\lambda_1 \notin \sigma(Z)$ by assumption), and $X := \sum_{j=1}^n A_{1,j}$, we obtain that

$$\sum_{j=1}^n \lambda_j P_j + \sum_{i,j=1}^n A_{i,j} \sim \sum_{j=1}^n \lambda_j P_j + \sum_{i,j=2}^n A_{i,j}.$$

The result then proceeds by induction by considering the unital C*-algebra $(I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$. \square

For convenience we make the following definition.

Definition 4.7. Let \mathfrak{A} be a unital C*-algebra. An operator $A \in \mathfrak{A}$ is said to be a scalar matrix in \mathfrak{A} if there exists a finite dimensional C*-algebra \mathfrak{B} and a unital, injective *-homomorphism $\pi : \mathfrak{B} \rightarrow \mathfrak{A}$ such that $A \in \pi(\mathfrak{B})$.

Remarks 4.8. Recall [Sk, Theorem 2.8] proves that if \mathfrak{A} is a unital, simple, purely infinite C*-algebra and $N \in \mathfrak{A}$ is a normal operator with $\sigma(N)$ connected, $0 \in \sigma(N)$, and $\lambda I - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$ then N is a norm limits of nilpotent scalar matrices in \mathfrak{A} . However, for the subsequent discussions, we will only need the fact that every normal operator with spectrum equal to the closed unit disk in the C*-algebras under our consideration is the norm limit of nilpotent scalar matrices. This is easily obtained based on Berg's Technique and will enable an alternate (yet more complicated) proof of [Sk, Theorem 2.8].

Proposition 4.9. Let \mathfrak{A} be a unital C*-algebra with the three properties listed in Theorem 4.2 or a unital, simple, purely infinite C*-algebra such that there exists an $n \in \mathbb{N}$ with $n[I_{\mathfrak{A}}]_0 = 0$. If $N \in \mathfrak{A}$ is a normal operator with the closed unit disk as spectrum then N is a norm limit of nilpotent scalar matrices from \mathfrak{A} .

Proof. Under the assumptions that \mathfrak{A} is a unital C*-algebra with the three properties listed in Theorem 4.2, it is easy to see the second and third assumptions imply that the 2^∞ -UHF C*-algebra has a unital, faithful embedding into \mathfrak{A} . In the case that \mathfrak{A} is a unital, simple, purely infinite C*-algebra, the condition $n[I_{\mathfrak{A}}]_0 = 0$ implies that the n^∞ -UHF C*-algebra has a unital, faithful embedding into \mathfrak{A} . Therefore, by [Sk, Theorem 6.10], \mathfrak{A} has a normal operator N_0 with the closed unit disk as spectrum that is a norm limit of nilpotent scalar matrices from \mathfrak{A} . Since every two normal operators with spectrum equal to the closed unit disk are approximately unitarily equivalent by Corollary 2.7 in the first setting and by Lemma 2.11 in the second setting, and since a unitary conjugate of a nilpotent scalar matrix is a nilpotent scalar matrix, the result follows. \square

For the proof of Theorem 4.1, we require the following technical result which emphasizes the condition that $n[I_{\mathfrak{A}}]_0 = 0$.

Lemma 4.10. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra such that there exists an $n \in \mathbb{N}$ with $n[I_{\mathfrak{A}}]_0 = 0$. Then there exists a unital, injective $*$ -homomorphism $\pi : \mathfrak{A}^{\oplus(n+1)} \rightarrow \mathfrak{A}$ such that if P_1, P_2, \dots, P_{n+1} are projections in \mathfrak{A} then $[\pi(P_1, \dots, P_{n+1})]_0 = \sum_{j=1}^{n+1} [P_j]_0$ (where $[0]_0 = 0$ for notational purposes).*

Proof. Since \mathfrak{A} is a unital, simple, purely infinite C^* -algebra, there exists n isometries $(V_k)_{k=1}^n$ such that the projections $P_k := V_k V_k^*$ are mutually orthogonal and $\sum_{k=1}^n P_k < I_{\mathfrak{A}}$. Let $P_{n+1} := I_{\mathfrak{A}} - \sum_{k=1}^n P_k$. Then

$$[P_{n+1}]_0 = [I_{\mathfrak{A}}]_0 - \sum_{k=1}^n [P_k]_0 = [I_{\mathfrak{A}}]_0 - n[I_{\mathfrak{A}}]_0 = [I_{\mathfrak{A}}]_0.$$

Therefore [Cu, Theorem 1.4] implies that P_{n+1} and $I_{\mathfrak{A}}$ are Murray-von Neumann equivalent. Hence there exists an isometry $V_{n+1} \in \mathfrak{A}$ such that $V_{n+1} V_{n+1}^* = P_{n+1}$.

Define $\pi : \mathfrak{A}^{\oplus(n+1)} \rightarrow \mathfrak{A}$ by

$$\pi(A_1, \dots, A_{n+1}) = \sum_{j=1}^{n+1} V_j A_j V_j^*.$$

It is trivial to verify that π is a unital, injective $*$ -homomorphism. The final statement of the lemma follows from the fact that if $P \in \mathfrak{A}$ is a projection then VPV^* and P are Murray-von Neumann equivalent for every isometry $V \in \mathfrak{A}$. \square

The following is our first real use of nilpotent scalar matrices and the main stepping-stone for Theorem 4.1. The idea is based on [He, Lemma 5.3].

Lemma 4.11. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra such that there exists an $n \in \mathbb{N}$ with $n[P]_0 = 0$ for all non-zero projections $P \in \mathfrak{A}$ and let $\pi : \mathfrak{A}^{\oplus(n+1)} \rightarrow \mathfrak{A}$ be the unital, injective $*$ -homomorphism from Lemma 4.10. Let $M \in \mathfrak{A}$ be a normal operator with $\lambda I_{\mathfrak{A}} - M \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$, let $\mu \in \sigma(M)$ be a cluster point of $\sigma(M)$, and let $Q \in \mathfrak{A}$ be a nilpotent scalar matrix. Then $M \oplus (\bigoplus_{k=1}^n (\mu I_{\mathfrak{A}} + Q)) \in \overline{\mathcal{S}(M)}$.*

Proof. Since $Q \in \mathfrak{A}$ is a nilpotent scalar matrix, Q is unitarily equivalent to a strictly upper triangular scalar matrix. Thus we can assume $Q = \bigoplus_{k=1}^m A_k$ where $A_k \in \mathcal{M}_{n_k}(\mathbb{C})$ are strictly upper triangular matrices. Let $\ell := \sum_{k=1}^m n_k$. Since $\mu \in \sigma(M)$ is a cluster point of $\sigma(M)$, there exists a sequence $(\mu_j)_{j \geq 1} \subseteq \sigma(M)$ of distinct scalars converging to μ . For each $q \in \mathbb{N}$ let

$$T_q := \text{diag}(\mu_q, \mu_{q+1}, \dots, \mu_{q+\ell-1}) \in \bigoplus_{k=1}^m \mathcal{M}_{n_k}(\mathbb{C}) \subseteq \mathfrak{A}$$

be the diagonal matrix with $\mu_q, \dots, \mu_{q+\ell-1}$ along the diagonal (technically T_q is a direct sum of diagonal matrices).

Let $M_q := M \oplus (\bigoplus_{k=1}^n T_q) \in \mathfrak{A}$. Then M_q is a normal operator with $\sigma(M_q) = \sigma(M)$. Moreover, since $\lambda I_{\mathfrak{A}} - M \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$ and π is a unital, injective $*$ -homomorphism, it is clear that $\lambda I_{\mathfrak{A}} - M_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$. In addition, if $f : \mathbb{C} \rightarrow \mathbb{C}$ is a function that is analytic on an open neighbourhood U of $\sigma(M)$ with $f(U) \subseteq \{0, 1\}$ then

$$f(M_q) = f(M) \oplus \left(\bigoplus_{k=1}^n f(T_q) \right)$$

so either $f(M) = 0$ so $f(M_q) = 0$ or

$$[f(M_q)]_0 = [f(M)]_0 + n[f(T_q)]_0 = [f(M)]_0$$

by Lemma 4.10 and the assumption that $n[P]_0 = 0$ for all non-zero projections $P \in \mathfrak{A}$. Hence $f(M_q)$ and $f(M)$ are Murray-von Neumann equivalent by [Cu, Theorem 1.4]. Therefore Theorem 2.15 implies $M \sim_{au} M_q$. Hence

$$M \sim_{au} M_q \sim M \oplus \left(\bigoplus_{k=1}^n T_q + Q \right)$$

by Lemma 4.5. Since $\lim_{q \rightarrow \infty} T_q + Q = \mu I + Q$, the result follows. \square

Using the ideas of the above proof, it is possible to prove the following.

Lemma 4.12. *Let \mathfrak{A} be a unital C^* -algebra with the following conditions;*

- (1) *there exists a unital, injective $*$ -homomorphism $\pi : \mathfrak{A} \oplus \mathfrak{A} \rightarrow \mathfrak{A}$, and*
- (2) *if $N_1, N_2 \in \mathfrak{A}$ are normal operators with $\lambda I_{\mathfrak{A}} - N_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N_q)$ and $q \in \{1, 2\}$, $N_1 \sim_{au} N_2$ if and only if $\sigma(N_1) = \sigma(N_2)$.*

Let $M \in \mathfrak{A}$ be a normal operator with $\lambda I_{\mathfrak{A}} - M \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$, let $\mu \in \sigma(M)$ be a cluster point of $\sigma(M)$, and let $Q \in \mathfrak{A}$ be a nilpotent scalar matrix. Then $M \oplus (\mu I + Q) \in \overline{\mathcal{S}(M)}$.

Subsequently we have our next stepping-stone which based on [He, Corollary 5.5].

Lemma 4.13. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra such that there exists an $n \in \mathbb{N}$ with $n[P]_0 = 0$ for all non-zero projections $P \in \mathfrak{A}$. Let $N, M \in \mathfrak{A}$ be normal operators and write $\sigma(N) = K_1 \cup K_2$ where K_1 and K_2 are disjoint compact sets with K_1 connected. Suppose*

- (1) $\sigma(M) = K'_1 \cup K_2$ where $K'_1 \subseteq K_1$,
- (2) $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$,
- (3) $\lambda I_{\mathfrak{A}} - M \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$, and
- (4) *for every function $f : \mathbb{C} \rightarrow \mathbb{C}$ that is analytic on an open neighbourhood U of $\sigma(N)$ with $f(U) \subseteq \{0, 1\}$, the projections $f(N)$ and $f(M)$ are Murray-von Neumann equivalent.*

If K'_1 contains a cluster point of $\sigma(M)$ then $N \in \overline{\mathcal{S}(M)}$.

Proof. If K_1 is a singleton, $K'_1 = K_1$ as K'_1 is non-empty. Thus $\sigma(M) = \sigma(N)$ so Theorem 2.15 implies N and M are approximately unitarily equivalent.

Otherwise K'_1 is not a singleton. Let $\epsilon > 0$, let $T \in \mathfrak{A}$ be a normal operator with

$$\sigma(T) = \{z \in \mathbb{C} \mid |z| \leq \epsilon\},$$

and let $\mu \in K'_1$ be a cluster point of $\sigma(M)$. Lemma 4.11 implies that $M \oplus (\bigoplus_{k=1}^n (\mu I_{\mathfrak{A}} + Q)) \in \overline{\mathcal{S}(M)}$ for every nilpotent scalar matrix $Q \in \mathfrak{A}$. Since T is a norm limit of nilpotent scalar matrices of \mathfrak{A} by Proposition 4.9, we obtain that $M \oplus (\bigoplus_{k=1}^n (\mu I_{\mathfrak{A}} + T)) \in \overline{\mathcal{S}(M)}$.

Let $M_1 := M \oplus (\bigoplus_{k=1}^n (\mu I_{\mathfrak{A}} + T))$ which is clearly a normal operator. Notice that $\lambda I_{\mathfrak{A}} - M_1 \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M_1)$. Moreover, if $f : \mathbb{C} \rightarrow \mathbb{C}$ is a function that is analytic on an open neighbourhood U of $\sigma(N) \cup \sigma(M_1)$ with $f(U) \subseteq \{0, 1\}$ then either $f(M) = 0$ so $f(M_1) = 0$ and $f(N) = 0$, or

$$[f(M_1)]_0 = [f(M)]_0 + n[f(\mu I_{\mathfrak{A}} + T)]_0 = [f(M)]_0 = [f(N)]_0$$

by Lemma 4.10 and the assumption that $n[P]_0 = 0$ for all non-zero projections $P \in \mathfrak{A}$. Hence $f(M_1)$ and $f(N)$ are Murray-von Neumann equivalent by [Cu, Theorem 1.4].

Since K_1 is connected and $\sigma(M_1)$ contains an open neighbourhood around $\mu \in K_1$, we can repeat the above argument a finite number of times to obtain a normal operator $M_0 \in \overline{\mathcal{S}(M)}$ such that $\sigma(M_0) = K_1'' \cup K_2$ where $K_1 \subseteq K_1''$,

$$K_1'' \subseteq \{z \in \mathbb{C} \mid \text{dist}(z, K_1) \leq \epsilon\},$$

$\lambda I_{\mathfrak{A}} - M_0 \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M_0)$, and if $f : \mathbb{C} \rightarrow \mathbb{C}$ is a function that is analytic on an open neighbourhood U of $\sigma(M_0)$ with $f(U) \subseteq \{0, 1\}$ then $f(M_0)$ and $f(N)$ are Murray-von Neumann equivalent. Therefore, Corollary 3.8 implies $\text{dist}(N, \mathcal{U}(M_0)) \leq \epsilon$ so $\text{dist}(N, \mathcal{S}(M)) \leq \epsilon$. Thus, as $\epsilon > 0$ was arbitrary, the result follows. \square

We can now complete the proof of Theorem 4.1 using the above result.

Proof of Theorem 4.1. Let N and M satisfy the five conditions of Theorem 4.1. By applying Lemma 4.14 recursively a finite number of times, we can find a normal operator M' such that $M' \in \overline{\mathcal{S}(M)}$, the spectrum of M' is $\sigma(M)$ unioned with finitely many connected components of $\sigma(N)$ (so $\sigma(M') \subseteq \sigma(N)$), and if $f : \mathbb{C} \rightarrow \mathbb{C}$ is a function that is analytic on an open neighbourhood U of $\sigma(N)$ with $f(U) \subseteq \{0, 1\}$ then $f(M')$ and $f(N)$ are Murray-von Neumann equivalent.

Fix $\epsilon > 0$. Since $\sigma(N)$ is compact, $\sigma(N)$ has a finite ϵ -net. Thus the normal operator M' in the above paragraph can be selected with the additional requirement that $\text{dist}(\lambda, \sigma(M')) \leq 2\epsilon$ for all $\lambda \in \sigma(N)$. By Corollary 3.8, $\text{dist}(N, \mathcal{U}(M')) \leq 2\epsilon$ so $\text{dist}(N, \mathcal{S}(M)) \leq 2\epsilon$ as desired. \square

By using similar ideas to the proof of Theorem 4.1 and by using the following lemma, the proof of Theorem 4.2 is also complete.

Lemma 4.14. *Let \mathfrak{A} be a unital C^* -algebra with the three properties listed in Theorem 4.2. Let $N, M \in \mathfrak{A}$ be normal operators with $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$ and $\lambda I_{\mathfrak{A}} - M \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$. Let $\{K_\lambda\}_\Lambda$ be the connected components of $\sigma(N)$. Suppose*

$$\sigma(M) = \left(\bigcup_{\lambda \in \Lambda \setminus \{\lambda_0\}} K_\lambda \right) \cup K_0$$

where $K_0 \subseteq K_{\lambda_0}$. If K_0 contains a cluster point of $\sigma(M)$ then $N \in \overline{\mathcal{S}(M)}$.

Proof. The proof of this lemma follows the proof of Lemma 4.13 by an application of Proposition 3.3 provided we can apply Lemma 4.12. Note that the second and third assumptions of Theorem 4.2 imply that the first assumption of Lemma 4.12 holds and Corollary 2.7 implies that the second assumption of Lemma 4.12 holds. \square

Based on Theorem 1.5, Proposition 2.10, Theorem 4.1, and Remarks 4.3, we raise the following question.

Question 4.15. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra and let $N, M \in \mathfrak{A}$ be normal operators. Is $N \in \overline{\mathcal{S}(M)}$ if and only if*

- (1) $\sigma(M) \subseteq \sigma(N)$,
- (2) $\sigma(M) \cap K_\lambda \neq \emptyset$ for all $\lambda \in \Lambda$,

- (3) $\lambda I_{\mathfrak{A}} - N$ and $\lambda I_{\mathfrak{A}} - M$ are in the same connected component of the identity for all $\lambda \notin \sigma(N)$,
- (4) $\sigma(M) \cap K_{\lambda}$ contains a cluster point of $\sigma(M)$ whenever K_{λ} is not a singleton, and
- (5) for every function $f : \mathbb{C} \rightarrow \mathbb{C}$ that is analytic on an open neighbourhood U of $\sigma(N)$ with $f(U) \subseteq \{0, 1\}$, the projections $f(N)$ and $f(M)$ are Murray-von Neumann equivalent.

With the proofs of Theorem 4.1 and Theorem 4.2 complete, we will classify when a normal operator in a C*-algebra satisfying the assumptions of Theorem 4.2 is a norm limit of nilpotent operators. This proof is based on the proof of [He, Proposition 5.6]. Furthermore this result has slightly weaker conditions to any result given in [Sk] (that is, there should exist C*-algebras satisfying the assumptions of the following theorem that are not studied in [Sk] although the author is not aware of them). However, we note the proof of [Sk, Theorem 2.8] can be adapted to this setting.

Corollary 4.16. *Let \mathfrak{A} be a unital, separable C*-algebra with the three properties listed in Theorem 4.2. A normal operator $N \in \mathfrak{A}$ is a norm limit of nilpotent operators from \mathfrak{A} if and only if $0 \in \sigma(N)$, $\sigma(N)$ is connected, and $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$.*

Proof. The requirements that $\sigma(N)$ is connected and contains zero follows by [Sk, Lemma 1.3]. The condition that $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$ follows by [Sk, Lemma 2.7].

Suppose $N \in \mathfrak{A}$ is a normal operator such that $0 \in \sigma(N)$, $\sigma(N)$ is connected, and $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$. Let $\epsilon > 0$ and fix a normal operator $T \in \mathfrak{A}$ such that

$$\sigma(T) = \{z \in \mathbb{C} \mid |z| \leq \epsilon\}.$$

The second and third assumptions of Theorem 4.2 implies that there exists a unital inclusion of $\mathfrak{A} \oplus \mathfrak{A}$ inside \mathfrak{A} . Let $M := N \oplus T$. Clearly M is a normal operator such that $\lambda I_{\mathfrak{A}} - M \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$. Note Proposition 3.3 implies

$$\text{dist}(N, \mathcal{U}(M)) \leq \epsilon.$$

However, as $\sigma(N)$ is connected and contains zero, $\sigma(M)$ is connected and contains $\sigma(T)$. Thus Theorem 4.2 implies that $M \in \overline{\mathcal{S}(T)}$ so

$$\text{dist}(N, \mathcal{S}(T)) \leq \epsilon.$$

However Proposition 4.9 implies that T is a norm limit of nilpotent operators from \mathfrak{A} and thus the above inequality implies N is within 2ϵ of a nilpotent operator from \mathfrak{A} . \square

Since Proposition 4.9 was used instead of [Sk, Theorem 2.8], a similar argument using Theorem 4.1 can be given to classify which normal operators in a unital, simple, purely infinite C*-algebra are a limit of nilpotent operators provided that there exists an $n \in \mathbb{N}$ with $n[P]_0 = 0$ for all non-zero projections $P \in \mathfrak{A}$. This provides an alternate (yet more difficult) proof of [Sk, Theorem 2.8] in this context.

It is natural to ask whether the constraint ‘there exists an $n \in \mathbb{N}$ such that $n[P]_0 = 0$ for all projections $P \in \mathfrak{A}$ ’ can be removed from the suppositions of Theorem 4.1. Unfortunately the complete classification has not been obtained in this setting. The main issue in the adaptation of Lemma 4.11 to this context is an

issue with the application of Theorem 2.15. However, the following result is almost a complete classification.

Proposition 4.17. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra and let $N, M \in \mathfrak{A}$ be normal operators such that $\lambda I_{\mathfrak{A}} - M \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$. Let $\{K_\lambda\}_\Lambda$ be the connected components of $\sigma(N)$. If*

- (1) $\sigma(M) \subseteq \sigma(N)$,
- (2) $\sigma(M) \cap K_\lambda \neq \emptyset$ for all $\lambda \in \Lambda$,
- (3) $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$,
- (4) $\sigma(M) \cap K_\lambda$ contains a connected component $\sigma(M)$ that is not a singleton whenever K_λ is not a singleton, and
- (5) for every function $f : \mathbb{C} \rightarrow \mathbb{C}$ that is analytic on an open neighbourhood U of $\sigma(N)$ with $f(U) \subseteq \{0, 1\}$, the projections $f(N)$ and $f(M)$ are Murray-von Neumann equivalent,

then $N \in \overline{\mathcal{S}(M)}$.

The main proof of Proposition 4.17 will follow the proof of Theorem 4.1 provided we can generalize Lemma 4.11 and Lemma 4.13. Thus the proof will be complete with the following two lemmas.

Lemma 4.18. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra, let $V \in \mathfrak{A}$ be an isometry, let $P := VV^*$, and let $M \in \mathfrak{A}$ be a normal operator with $\lambda I_{\mathfrak{A}} - M \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$. Suppose μ is an element of a connected component of $\sigma(M)$ that is not a singleton and $Q \in (I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$ is a nilpotent scalar matrix. Then $VMV^* + \mu(I_{\mathfrak{A}} - P) + Q \in \overline{\mathcal{S}(M)}$.*

Proof. The main difference between the following proof and that of Lemma 4.11 comes from the lack of additional structure of the projections in our C^* -algebra \mathfrak{A} . Since $Q \in (I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$ is a nilpotent scalar matrix, Q is unitarily equivalent to a strictly upper triangular scalar matrix. Thus we can assume $Q = \oplus_{k=1}^m A_k$ where $A_k \in \mathcal{M}_{n_k}(\mathbb{C})$ are strictly upper triangular matrices. Let $\ell := \sum_{k=1}^m n_k$. By our assumptions on μ , there exists a sequence $(\mu_j)_{j \geq 1}$ of distinct scalars contained in the connected component of $\sigma(M)$ containing μ that converges to μ . For each $q \in \mathbb{N}$ let

$$T_q := \text{diag}(\mu_q, \mu_{q+1}, \dots, \mu_{q+\ell-1}) \in \bigoplus_{k=1}^m \mathcal{M}_{n_k}(\mathbb{C}) \subseteq (I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$$

be the diagonal matrix with $\mu_q, \dots, \mu_{q+\ell-1}$ along the diagonal.

Let $M_q := VMV^* + T_q \in \mathfrak{A}$. Since V is an isometry, M_q is a normal operator. Moreover, it is easy to verify that $\sigma(M_q) = \sigma(M)$ and, since $\lambda I_{\mathfrak{A}} - M \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$, $\lambda I_{\mathfrak{A}} - M_q \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$. Furthermore, if $f : \mathbb{C} \rightarrow \mathbb{C}$ is a function that is analytic on an open neighbourhood U of $\sigma(M)$ with $f(U) \subseteq \{0, 1\}$ then

$$f(M_q) = f(VMV^*) + f(T_q) = Vf(M)V^* + f(T_q).$$

Therefore, either $f(T_q) = 0$ so $f(M_q) = Vf(M)V^*$ is Murray-von Neumann equivalent to $f(M)$ or $f(T_q) = I_{\mathfrak{A}} - P$ as $\sigma(T_q)$ is contained in a connected component of $\sigma(M)$ so

$$[f(M_q)]_0 = [f(M)]_0 + [f(T_q)]_0 = [f(M)]_0$$

as $[I_{\mathfrak{A}} - P]_0$ is the identity element of $K_0(\mathfrak{A})$ by the proof of [Cu, Theorem 1.4]. Hence, in either case, $f(M_q)$ and $f(M)$ are Murray-von Neumann equivalent by [Cu, Theorem 1.4]. Therefore Theorem 2.15 implies $M \sim_{au} M_q$. Hence

$$M \sim_{au} M_q \sim M \oplus \left(\bigoplus_{k=1}^n T_q + Q \right)$$

by Lemma 4.5. Since $\lim_{q \rightarrow \infty} T_q + Q = \mu I + Q$, the result follows. \square

Lemma 4.19. *Let \mathfrak{A} be a unital, simple, purely infinite C^* -algebra. Let $N, M \in \mathfrak{A}$ be normal operators and write $\sigma(N) = K_1 \cup K_2$ where K_1 and K_2 are disjoint compact sets with K_1 connected. Suppose*

- (1) $\sigma(M) = K'_1 \cup K_2$ where $K'_1 \subseteq K_1$,
- (2) $\lambda I_{\mathfrak{A}} - N \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(N)$,
- (3) $\lambda I_{\mathfrak{A}} - M \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M)$, and
- (4) *for every function $f : \mathbb{C} \rightarrow \mathbb{C}$ is analytic on an open neighbourhood U of $\sigma(N)$ with $f(U) \subseteq \{0, 1\}$, the projections $f(N)$ and $f(M)$ are Murray-von Neumann equivalent.*

If K'_1 contains a connected component of $\sigma(M)$ with at least two points then $N \in \overline{\mathcal{S}(M)}$.

Proof. Let $\epsilon > 0$, let $V \in \mathfrak{A}$ be any proper isometry, let $P := VV^*$, let $T \in (I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$ be a normal operator with

$$\sigma(T) = \{z \in \mathbb{C} \mid |z| \leq \epsilon\},$$

and let $\mu \in K'_1$ be an element of a connected component of $\sigma(M)$ with at least two points. Lemma 4.18 implies that $VMV^* + \mu(I_{\mathfrak{A}} - P) + Q \in \overline{\mathcal{S}(M)}$ for every nilpotent scalar matrix $Q \in (I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$. Since $(I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$ is a unital, simple, purely infinite C^* -algebra, T is a norm limit of nilpotent scalar matrices of $(I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$ by [Sk, Theorem 2.8]. Hence $VMV^* + \mu(I_{\mathfrak{A}} - P) + T \in \overline{\mathcal{S}(M)}$.

Let $M_1 := VMV^* + \mu(I_{\mathfrak{A}} - P) + T$ which is clearly a normal operator. Notice that $\lambda I_{\mathfrak{A}} - M_1 \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M_1)$. Moreover, if $f : \mathbb{C} \rightarrow \mathbb{C}$ is a function that is analytic on an open neighbourhood U of $\sigma(N) \cup \sigma(M_1)$ with $f(U) \subseteq \{0, 1\}$ then either $f(M) = 0$ so $f(M_1) = 0$ and $f(N) = 0$ or

$$[f(M_1)]_0 = [f(VMV^*)]_0 = [Vf(M)V^*]_0 = [f(M)]_0 = [f(N)]_0$$

as either $f(\mu(I_{\mathfrak{A}} - P) + T)$ is zero or $I_{\mathfrak{A}} - P$ which has trivial K_0 -class. Hence $f(M_1)$ and $f(N)$ are Murray-von Neumann equivalent by [Cu, Theorem 1.4].

Since K_1 is connected and $\sigma(M_1)$ contains an open neighbourhood around $\lambda \in K_1$, we can repeat the above argument a finite number of times to obtain a normal operator $M_0 \in \overline{\mathcal{S}(M)}$ such that $\sigma(M_0) = K'_1 \cup K_2$ where $K_1 \subseteq K'_1$,

$$K'_1 \subseteq \{z \in \mathbb{C} \mid \text{dist}(z, K_1) \leq \epsilon\},$$

$\lambda I_{\mathfrak{A}} - M_0 \in \mathfrak{A}_0^{-1}$ for all $\lambda \notin \sigma(M_0)$, and if $f : \mathbb{C} \rightarrow \mathbb{C}$ is a function that is analytic on an open neighbourhood U of $\sigma(M_0)$ with $f(U) \subseteq \{0, 1\}$ then $f(M_0)$ and $f(N)$ are Murray-von Neumann equivalent. Therefore, Corollary 3.8 implies $\text{dist}(N, \mathcal{U}(M_0)) \leq \epsilon$ so $\text{dist}(N, \mathcal{S}(M)) \leq \epsilon$. Thus, as $\epsilon > 0$ was arbitrary, the result follows. \square

To conclude this paper, we will briefly discuss closed similarity orbits of normal operators in von Neumann algebras. We recall that [Sh] completely classifies when two normal operators are approximately unitarily equivalent in von Neumann algebras. Moreover, Theorem 4.2 completely determines when one normal operator is in the closed similarity orbit of another normal operator in type III factors with separable predual. Thus it is natural to ask whether a generalization of Theorem 4.2 to type II factors may be obtained.

However, the existence of a faithful, normal, tracial state on type II_1 factors inhibits when a normal operator can be in the closed similarity orbit of another normal operator. Indeed suppose \mathfrak{M} is a type II_1 factor and let τ be the faithful, normal, tracial state on \mathfrak{M} . If $N, M \in \mathfrak{M}$ are such that $N \in \overline{\mathcal{S}(M)}$, it is trivial to verify that $\tau(p(N)) = \tau(p(M))$ for all polynomials p in one variable. In particular, if $N, M \in \mathfrak{M}$ are self-adjoint and $N \in \overline{\mathcal{S}(M)}$, we obtain that $\tau(f(N)) = \tau(f(M))$ for all continuous functions on $\sigma(N) \cup \sigma(M)$ and, as τ is faithful and normal, this implies that N and M must have the same spectral distribution. Therefore, if $N, M \in \mathfrak{M}$ are self-adjoint operators, $\sigma(M) = [0, \frac{1}{2}]$, and $\sigma(N) = [0, 1]$, then, unlike in $\mathcal{B}(\mathcal{H})$, $N \notin \overline{\mathcal{S}(M)}$. Moreover, combining the above arguments and [Sh, Theorem 1.3], we have the following result.

Proposition 4.20. *Let \mathfrak{M} be a type II_1 factor. If $N, M \in \mathfrak{M}$ are self-adjoint operators and $N \in \overline{\mathcal{S}(M)}$, then $N \sim_{au} M$.*

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DEPARTMENT OF MATHEMATICS, UCLA, LOS ANGELES, CALIFORNIA, 90095-1555, USA
E-mail address: `pskoufra@math.ucla.edu`